Mortgage, Financial Intermediation and Optimism

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December 15, 2016

Abstract

This paper presents a new framework to study the macroeconomic effect of mortgage default, where the mortgage contracts are endogenously chosen. Our model shows that, financial intermediation provides risk-sharing and Pareto improves allocation in the housing market through the mortgage loan contracts. However, when a negative endowment shock occurs to the house buyers, the decreased house price triggers mortgage default, leading to less credit available from the bank for new mortgage loans. Decreased demand and fire sale by the bank further decreases house price, and a downward spiral of credit and prices materializes, which explain the transmission and amplification of the initial shock. A Bayesian learning banker becomes more pessimistic in bad times, amplifying the financial stress in the housing market.

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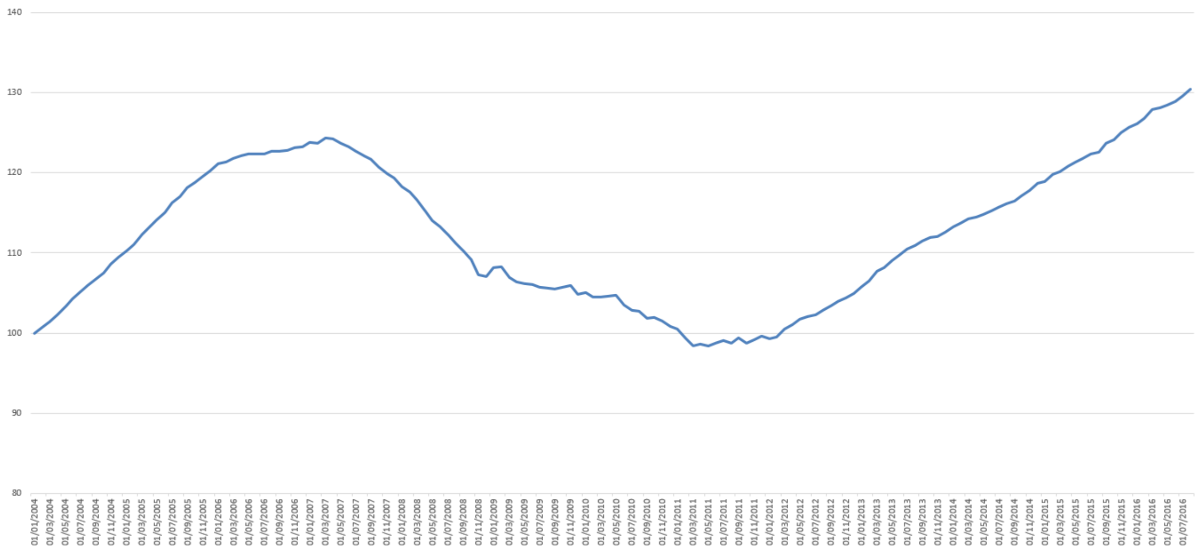
1 Introduction

The global financial crisis of 2007-2009 is considered the worst since the Great Depression in the 1930s. During the crisis, the burst of the U.S. housing bubble caused the values of securities tied to U.S. real estate pricing to plummet, damaging the financial institutions globally. However, the mainstream DSGE models fail to incorporate the possibility of default, thus are not powerful enough to either prediction the crisis, or give valuable policy implications. This paper introduces general equilibrium models with collateral and default to study the mortgage market and the role of financial intermediation. The models show that, the financial intermediation provides risk-sharing and Pareto improves allocation in the housing market by issuing mortgage loan contracts. When a negative shock hits the house buyers' endowment, the decreased house price triggers mortgage default, leading to less credit available from the bank for new mortgage loans. Decreased demand and fire sale by the bank further decreases house price, and a downward spiral of credit and prices materializes. A Bayesian learning banker becomes more pessimistic in bad times, amplifying the financial stress in the housing market.

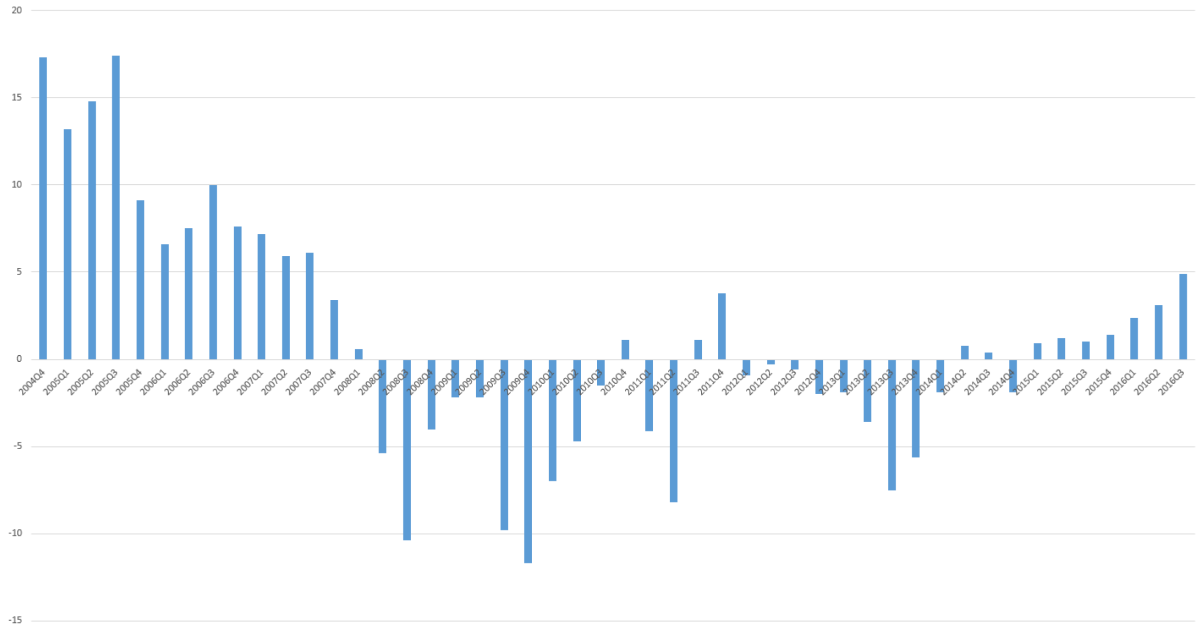
Figure 1: House Price Index



Monthly House Price index for U.S., purchase-only index, seasonally adjusted. Index at Jan. 2004 is normalized to 100. Data source: Federal Housing Finance Agency.

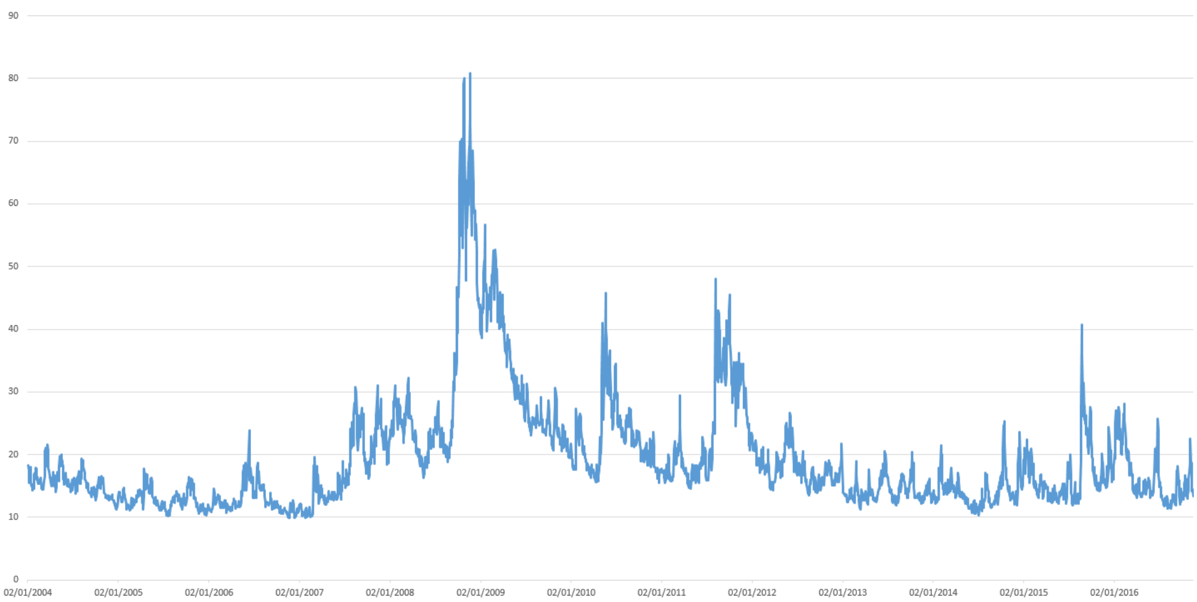
Figure 1 shows the monthly House Price Index (HPI) for U.S. from Jan. 2004 to July 2016, with the data for Jan. 2004 normalized to 100. The house price kept increasing before the mortgage crisis, until reaching the peak around March to April 2007, then began to slide down significantly since Oct. 2007. The decreasing trend did not stop until 2011. Figure 2 shows the changes in the U.S. residential real estate loans from commercial banks, from Q4 2004 to Q3 2016. Obviously, there is a sharp decrease in change of the loan extension in Q4 2007, which becomes almost zero in Q1 2008, and negative in Q2 2008. The residential real estate loans' extension did not stop decreasing until Q1 2015. Figure 3 presents the VIX index from Jan. 2004 to Nov. 2016. Although challenged and criticized, VIX is still a good measure for market fear or anxiety. The index started to fluctuate more since later 2007, and was boosted in Sep. 2008, reaching the peak around Oct. to Nov. 2008.

Figure 2: Residential real estate loans, percentage change per year



Percentage change per year in U.S. residential real estate loans, all commercial banks, seasonally adjusted, annual growth rate (break adjusted). Data source: Federal Reserve Bank, economics research & data.

Figure 3: Volatility S&P 500 (VIX)



Data source: Yahoo Finance.

The change in the leverage may have played an important role as well. As documented by Fostel and Geanakoplos (2016), the leverage, prices and investment in the US housing market moved together during the crisis. These data imply some interesting relationship among those economic variables. When the demand in the housing market starts to drop, the house price decreases, which decreases the demand for mortgage loans. But why does this lead to an unanticipated crisis? We know that due to the low credit quality in those sub-prime mortgages, defaults occurred inevitably when the house price plummeted. And due to financial innovation, mainly securitization, the crisis in the mortgage market quickly spread into the whole financial sector. The anxiety in the market crystallized, persisting and amplifying the crisis. This paper aims at offering an coherent model to include house price, leverage, default, optimism, and financial intermediaries.

The paper first introduces a mortgage loan contract, and a collateral equilibrium with endogenous default. Due to lack of ability to commit, loans are possible only when backed by collaterals. The mortgage loan contract in this paper is similar to the collateralized security in Geanakoplos and Zame (2014). One unit of mortgage loan contract required one unit of house as collateral. The price of the contract, π, represents how much can be borrowed per contract. If the house price is P_0 when the mortgage is issued, the Loan-to-Value ratio is π/P_0. Consequently the margin is $1 - \pi/P_0$, and the loan's leverage is $1/(1 - \pi/P_0)$. The face value of the loan A is the required repayment for the loan. If the realized future house price is higher than A, the house buyer is optimal to repay the loan in full. Otherwise, the house buyer may default on the mortgage loan, give up the collateral, and purchase some new houses at the new equilibrium price. This paper offers a different method to choose the contract. In equilibrium, households agree on the price and face value of the contract, as well as the change of contract price with respect to the face value around the equilibrium. Using this method we are able to endogenously determine the price and leverage at the same time.

Direct contracting between the households may not always be possible, either due to lack of ability, or the fact that the endogenously chosen contract is not actively traded. Using a similar structure as in Goodhart et al. (2012, 2013), this paper models a banker who offers mortgage loan contracts to house buyers and deposit contracts to house seller. In equilibrium, the existence of the banker Pareto improves the allocation between the households, by providing better risk-sharing, and completing the asset market.

The dynamic interactions between credit limits and asset prices transmit the initial shock into a persistent and amplified fluctuation in a wider economy in longer terms. The paper extends the two-period binomial model with one more period, with overlapping generation households and a long-living banker. The initial negative endowment shock lowers the house price, and triggers some mortgage loan default. The banker's balance sheet shrinks accordingly, thus less credit is available for new mortgage loans. Meanwhile, the banker needs to resell the seized collateral in the house market where the demand is already low. The Shleifer and Vishny (1992) type fire sale further decreases the house price. The stressed financial condition also makes the banker require tighter terms on the mortgage loans, namely higher collateral requirement and interest rate. This decreases the collateral value of the house, thus the house price as well. The credit cycle similar to Kiyotaki and Moore (1997) together with the price cycle with fire sale reinforce each other, amplifying the initial shock and transmitting it into the wider economy.

We also consider the learning and belief formation of the banker. When the banker observes the realization of state, she updates her belief by the Bayes' rule. The occurrence of a good state makes the banker more optimistic, while a bad state makes her less optimistic. Minsky (1992) states that, over a period in which the economy does well, optimism changes, and agents tend to invest more in the riskier asset. The tendency to transform doing well into a speculative investment boom is the basic instability in the economy. Minsky's argument elaborates how optimism grows during a long time of prosperity, while this paper focus on the situation when the economy is going down. The model shows that the learning behaviour of the banker exacerbates the effect of the initial shock, since the banker becomes less optimistic, or even panic, and rationally over-reacts to the shock. Thus, policies aiming at preventing the deterioration of financial crisis should take the change in the sentiment into consideration.

1.1 Related literature

This paper is related to literature on collateral in general equilibrium framework. Dubey, Geanakoplos, and Shubik (2005) extends the standard GEI model with default and punishment by thinking of assets as pools. Their model encompasses a broad rage of adverse selection and signalling phenomena, and proves the existence of the refined equilibrium. Araujo, Kubler, and Schommer (2012) examines the effects of default and collateral on risk sharing, and demonstrates that the scarcity of collateralizable durable goods leads to large welfare losses. Regulation of collateral requirements never leads to a Pareto improvement for all agents, although it would often benefit a large group of agents. Geanakoplos and Zame (2014) defines and proves the existence of the collateral equilibrium, and shows that collateral requirements inhibit lending, limit borrowing, and distort consumption decisions. The shortage in supply creates incentive to create collateral by the authority and to stretch existing collateral by the market through financial innovation. However, the possibility of robust inefficiency is inescapable.

This paper is also related to literature on borrowing constraints, credit cycle and financial stability. Shleifer and Vishny (1992) identify the fire-sale mechanism in which a firm in financial distress sells assets when its industry peers are likely to be experiencing problems. In our model, the bank re-sells the collateralized houses when default happens in the mortgage loan market, which is similar to a fire-sale. Bernanke, Gertler, and Gilchrist (1999) develop a DSGE model which clarifies the role of credit market frictions in business fluctuations. The framework exhibits a financial accelerator, where endogenous developments in credit markets work to amplify and propagate shocks to the macroeconomy. Holmstrom and Tirole (1997) shows that when firms and financial intermediaries are capital constrained, capital tightening hits poorly capitalized firms the hardest, but that interest rate effects and the intensity of monitoring will depend on relative changes in the various components of capital. Kiyotaki and Moore (1997) construct a dynamic economy in which durable assets are both factors of production and collateral for loans, and shows that the dynamic interaction between credit limits and asset prices is a powerful transmission mechanism by which the effects of shocks persist, amplify and spill over to other sectors. Goodhart, Sunirand, and Tsomocos (2006) introduce a GEI model with heterogeneous commercial banks, capital requirements, money and default, and allow for the possibility of capital requirements' violation and consequent penalties. Brunnermeier and Pedersen (2009) link an asset's market liquidity and traders' funding liquidity, and show that under certain conditions, margins are destabilizing and market liquidity and funding liquidity are mutually reinforcing, leading to liquidity spirals. Their model explains why market liquidity can suddenly dry up, has commonality across securities, is related to volatility, and is subject to flight to quality and co-moves with the market.

The simultaneous endogenous determination of leverage and interest rate, and the co-movement of leverage level and asset price in our model relates to works by Geanakoplos etc. Geanakoplos (1997) first shows how one supply-equals-demand equation can determine leverage and interest under uncertainty. Geanakoplos (2003) argues that in normal times leverage and asset prices get too high, and in bad times, when the future is worse and more uncertain, leverage and asset prices get too low. Geanakoplos and Fostel (2008) suggests the spread of leverage cycle one asset class to other unrelated asset classes. There are other explanations for the co-movement of asset price and leverage. Adrian and Shin (2010) document evidence that there is a strongly positive relationship between changes in leverage and changes in balance sheet size. Financial intermediaries adjust their balance sheets actively, and doing so in such a way that leverage is high during booms and low during busts. So leverage is procyclical, and this affects volatility and particularly the price of risk.

The rest of the paper is organized as follows: Section 2 models a mortgage loan contract directly between the house buyers and sellers in a simple binomial framework. Section 3 adds a banker into the previous model, and shows that the banker Pareto improves the allocation when direct contracting is not achievable. Section 4 extends the model with one more period, with overlapping generation households and a long-living banker, and shows that a credit cycle and price cycle due to the shrinkage of banker's balance sheet transmit the initial shock in house buyer's endowment into the overall economy. Section 5 shows that a Bayesian learning banker over-reacts, and amplifies the effects of the initial shock, deteriorating the financial condition in the bad times. Section 6 concludes the paper.

2 A simple mortgage model

This section shows how a mortgage contract is chosen between a house seller and a house buyer. Assume at this stage that there is no financial intermediary, and the households can create mortgage contracts directly with each other.

2.1 The economy

Consider a two-period ($t = 0, 1$) binomial economy with two types of households, the house seller (R) and buyer (P), who are atomic and of equal mass. The house buyer is endowed with perishable consumption goods at each state, while the house seller is endowed with some durable houses at the beginning, and some consumption goods at each state as well. The house buyer's endowment is high if a good state occurs at $t = 1$, and low if a bad state occurs at $t = 1$. Both types of households derive utility from consuming the goods and living in the houses, and the house provides some salvage value in the end. Households trade with each other to maximize expected utility. There is no fiat money in the economy, and the consumption good is used as numeraire.

If there is a full set of contingent claims, and the household can commit to repay all the obligations, fist-best is achieved. However, assume that the asset market is incomplete, and the households repay their debt only when it is backed by some collateral, whose value is higher than the debt's face value at maturity. The lack of commitment creates the need for a mortgage contract.

Assume for now that the households can create mortgage loan contracts directly with each other, without the help of a financial intermediary. Following Geanakoplos and Zame (2014), a collateralized loan contract is defined as a pair of variables: (A, C), where A is the required repayment per contract, and C is the required collateral per contract. The contract's price, π, represents how much the house buyer can borrow from the seller per contract. From now on, normalize C to 1, which means each unit of mortgage loan contract requires one unit of house as collateral, and assume that the contract is infinitely divisible. If the future house price is higher than A, it is optimal for the borrower/buyer to repay the mortgage loan in full, otherwise default occurs, and the lender/seller seizes the collateralized houses. Partial default is not allowed, so the buyer either repay in full, or default on all her mortgage loans. The households are atomic, so they do not internalized the impact on the loan's price of their the choice of quantity in the mortgage loan contracts to use, nor do they realize their impact on the house price. However, both types of households choose the face value, and they do realize that contracts with different face values should have different prices. In equilibrium, they agree on the face value, as well as the impact of change in face value on the loan's price.

At the initial state ($t = 0$), the two types of households enter the economy. The buyer purchases the houses from the seller, and borrows at the same time, using the mortgage loan contract. At $t = 1$, if a good state (U) occurs, the buyer's new endowment is high, and the realized house price is high enough such that it is optimal for the buyer to fully repay the mortgage loans. The buyer may purchase some more houses from the seller. If a bad state occurs, the buyer's new endowment is low. If the endowment is so low that either the buyer cannot afford to repay the mortgage loan, or the realized house price is lower than the face value of the mortgage loan. The buyer would default on the mortgage loan, give up the collateral, and purchase some houses at the new equilibrium price.

2.2 The optimization problem

2.2.1 The lender/house seller (R)

The house seller is the lender in the mortgage loan contract, who derives utility from consuming the consumption goods and living in houses. The house also provides some salvage value in the end. Endowed with some houses (E) at the beginning and the consumption goods at each state ($e^R \equiv \{e_0^R, e_U^R, e_D^R\}$), the house seller chooses her consumption ($c^R \equiv \{c_0^R, c_U^R, c_D^R\}$), the quantity of houses to sell ($q^R \equiv \{q_0^R, q_U^R, q_D^R\}$), the numbers of mortgage loan contract to issue (ϕ^R), and the mortgage loan's face value (A^R) to maximize her expected utility,

$$\begin{aligned}\bar{U}^R =& U(c_0^R, E - q_0^R) + \beta^R \theta^R \left(U(c_U^R, E - q_0^R - q_U^R) + S(E - q_0^R - q_U^R)\right) \\ &+ \beta^R (1 - \theta^R) \left(U(c_D^R, E - q_0^R + \phi^R - q_D^R) + S(E - q_0^R + \phi^R - q_D^R)\right)\end{aligned} \tag{2.1}$$

subject to the budget set $\mathcal{B}_1^R(P, \pi)$ defined by inequalities (2.2)~(2.4), where $U(\cdot)$ is the utility function, $S(\cdot)$ is the salvage value of the houses, β^R is the discount factor, θ^R is the seller's subjective probability for state U to occur, and $P \equiv \{P_0, P_U, P_D\}$ are the house prices at each state. At the initial state, her consumption plus the lending should not exceed her endowment plus the income from selling the house,

$$c_0^R + \pi \phi^R \leqslant e_0^R + P_0 q_0^R \tag{2.2}$$

If state U occurs, her consumption should not exceed the sum of her endowment, income from selling some more houses, and the mortgage loan repayment,

$$c_U^R \leqslant e_U^R + P_U q_U^R + A^R \phi^R \tag{2.3}$$

If state D occurs, her consumption should not exceed her endowment, plus the income from selling some more houses.

$$c_D^R \leqslant e_D^R + P_D q_D^R \tag{2.4}$$

2.2.2 The borrower/house buyer (P)

The house buyer is the borrower in the mortgage loan contract, who also derives utility from consuming the consumption goods and living in houses. The house also provides some salvage value in the end. She is only endowed with some consumption goods at each state (e_0^P, e_U^P, e_D^P), and e_D^P is low. The buyer chooses her consumption (c_0^P, c_U^P, c_D^P), the quantity of houses to buy (q_0^P, q_U^P, q_D^P), the numbers of mortgage loan contract to use (ϕ^R), and the mortgage loan's face value (A^R) to maximize her expected utility,

$$\begin{aligned}\bar{U}^P =& U(c_0^P, q_0^P) + \beta^P \theta^P \left(U(c_U^P, q_0^P + q_U^P) + S(q_0^P + q_U^P)\right) \\ &+ \beta^P (1 - \theta^P) \left(U(c_D^P, q_0^P - \phi^P + q_D^P) + S(q_0^P - \phi^P + q_D^P)\right)\end{aligned} \tag{2.5}$$

subject to the budget set $\mathcal{B}_1^P(P, \pi)$ defined by inequalities (2.6)~(2.9). In the initial state, the buyer's consumption plus the payment for the houses should not exceed her endowment plus the borrowed amount,

$$c_0^P + P_0 q_0^P \leqslant e_0^P + \pi\phi^P \tag{2.6}$$

Meanwhile, the collateral constraint is,

$$\phi^P \leqslant q_0^P \tag{2.7}$$

If the good state (U) occurs, the sum of her consumption, payment for some new houses, and the mortgage loan repayment should not exceed her endowment,

$$c_U^P + P_U q_U^P + A^P\phi^P \leqslant e_U^P \tag{2.8}$$

If the bad state (D) occurs, the buyer defaults on the mortgage loan, and her consumption plus the payment for some new houses should not exceed her endowment,

$$c_D^P + P_D q_D^P \leqslant e_D^P \tag{2.9}$$

2.3 The equilibrium

In equilibrium the house markets and the mortgage loan market should all clear, namely the supply of the houses equals the demand for the houses at each state,

$$q_0^R = q_0^P \equiv q_0 \tag{2.10}$$

$$q_U^R = q_U^P \equiv q_U \tag{2.11}$$

$$q_D^R = q_D^P \equiv q_D \tag{2.12}$$

the mortgage loan contract's supply equals demand,

$$\phi^R = \phi^P \equiv \phi \tag{2.13}$$

the two types of households agree on the mortgage loan's face value,

$$A^R = A^P \equiv A \tag{2.14}$$

and that the households agree on the change in contract price with respect to face value around the equilibrium,

$$\frac{\partial\pi}{\partial A^R} = \frac{\partial\pi}{\partial A^P} \tag{2.15}$$

We need to determine which mortgage contract is positively traded in equilibrium, i.e. the face value A. The market decides the price π. Consequently the interested rate and the leverage ratio are determined. The problem is, one "supply-equals-demand" relationship needs to determine two variables simultaneously. Market clearing condition (2.15) implies that the households realize that mortgage contracts with different face value have different prices, and they agree on the change in contract price with respect to the change in face value around the equilibrium. More precisely, the interest rate of the contract is given by,

$$1 + r = \frac{A}{\pi} \tag{2.16}$$

Thus, $\frac{\partial \pi}{\partial A}$ can be written as,

$$\frac{\partial \pi}{\partial A} = \frac{\partial \frac{A}{1+r}}{\partial A} = \frac{1 + r - A\frac{\partial r}{\partial A}}{(1+r)^2} = \frac{1 - \frac{\partial(1+r)/(1+r)}{\partial A/A}}{1+r} = \frac{1 - \eta_{1+r,A}}{1+r} \tag{2.17}$$

where $\eta_{1+r,A}$ is the elasticity of the contract's required rate of return w.r.t. the face value. So the market clearing condition (2.15) requires that the households should agree on the interest rate, as well as the elasticity above.

Definition 1. *The Collateral Equilibrium for the economy*

$$E_1 = \left\{ (\bar{U}^h, e^h, E)_{h \in \{R,P\}} \right\}$$

is a vector consisting of house prices, mortgage loan contract price, consumptions, house trading, mortgage contract's face value and traded volume,

$$\Big((P, \pi), (c^h, q^h)_{h \in \{R,P\}}, A, \phi \Big)$$

if and only if

1. $(c^h, q^h, \phi^h, A^h) \in \mathcal{B}_1^h(P, \pi)$, $\forall h \in \{R, P\}$
2. $(\tilde{c}, \tilde{q}, \tilde{\phi}, \tilde{A}) \in \mathcal{B}_1^h(P, \pi) \Rightarrow \bar{U}^h(\tilde{c}, \tilde{q}) \leqslant \bar{U}^h(c^h, q^h)$, $\forall h \in \{R, P\}$
3. *Market clearing conditions (2.10)∼(2.15) hold.*
4. *In equilibrium* $\min\left[\frac{e_D^P}{\phi}, P_D\right] < A < \min\left[\frac{e_U^P}{\phi}, P_U\right]$

Note that the last condition implies that at state U, it is optimal and affordable for the house buyer to repay the mortgage loan, while at state D, it is either optimal for her to default, or the debt is unaffordable.

Proposition 1. *When the house is used as collateral in the mortgage loan, the price is higher than the fundamental value for the buyer. The difference represents the collateral value of the house.*

Proof. From Appendix A.1,

$$P_0 = \frac{U'(H_0^P) + \beta^P \theta^P (U'(H_U^P) + S'(H_U^P)) + \beta^P (1 - \theta^P)(U'(H_D^P) + S'(H_D^P))}{U'(c_0^P)} + \frac{\mu}{U'(c_0^P)}$$

where $H_s^P, s \in \{0, U, D\}$ is the house owed by P at state s. The first term represents the house's fundamental value to the house buyer, which is the sum of the marginal rates of substitution, and the second term represents the collateral value, with the KKT multiplier $\mu > 0$, and

$$\mu = \pi U'(c_0^P) - \theta A U'(c_U^P) - (1 - \theta) U'(H_D^P)$$

Combining the two equations we have,

$$(P_0 - \pi)U'(c_0^P) + \beta^P\theta^P AU'(c_U^R) = U'(q_0^P) + \beta^P\theta^P \left(U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P)\right)$$

which implies that the marginal cost of down payment plus the marginal cost of the repayment should equal to the marginal benefit from living in the house. These results are similar to those in Geanakoplos and Fostel (2008). □

Proposition 2. *If the two types of households have the same instantaneous utility function as*

$$U(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln(x) & \gamma = 1 \end{cases}$$

and the salvage value function $S(\cdot) = \tau U(\cdot)$, then the house price at state U and D are determined by the endowment as

$$P_U = (1+\tau)\left(\frac{e_U^R + e_U^P}{E}\right)^\gamma$$
$$P_D = (1+\tau)\left(\frac{e_D^R + e_D^P}{E}\right)^\gamma$$

where τ is the salvage value coefficient, and γ is the risk-aversion coefficient. Moreover, if the house sellers and buyers have the same discount factor(β) and same belief (θ), then no house is traded at state U.

Proof. The first order conditions are given in Appendix A.1. □

2.4 Numerical Example

Consider the exogenous variables in Table 1. The house seller is endowed with 1 unit of houses at the beginning, and her endowment in consumption goods are the same in state U and D. The house buyer's endowment is 20 in the good state, and only 5 in the bad state, which leads to default in equilibrium. Let the utility function be logarithm ($\gamma = 1$), and separable across time and goods,

$$\begin{aligned}\bar{U}^R &= \ln(c_0^R) + \ln(E - q_0^R) + \beta^R\theta^R\left(\ln(c_U^R) + (1+\tau)\ln(E - q_0^R - q_U^R)\right) \\ &\quad + \beta^R(1-\theta^R)\left(\ln(c_D^R) + (1+\tau)\ln(E - q_0^R + \phi - q_D^R)\right) \\ \bar{U}^P &= \ln(c_0^P) + \ln(q_0^P) + \beta^P\theta^P\left(\ln(c_U^P) + (1+\tau)\ln(q_0^P + q_U^P)\right) \\ &\quad + \beta^P(1-\theta^P)\left(\ln(c_D^P) + (1+\tau)\ln(q_0^P - \phi + q_D^P)\right)\end{aligned}$$

The endogenous variables are given in Table 2. In equilibrium, the house buyer's collateral constraint binds. The house buyer would like to borrow more, but is not able to do so due to lack of collateral. The marginal benefit of borrowing one more unit of mortgage loan is $\pi U'(c_0^P)$,

Table 1: Exogenous variables, two households two period binomial model

$E = 1$	$\theta = 0.99$	$\tau = 0.8$	$\beta^R = 1$	$\beta^P = 0.8$	
$e_0^R = 10$	$e_U^R = 20$	$e_D^R = 20$	$e_0^P = 10$	$e_U^P = 20$	$e_D^P = 5$

while the marginal cost is $\beta^P \theta^P A U'(c_U^P) + \beta^P (1-\theta^P)(1+\tau)U'(q_D^P)$. In equilibrium the marginal benefit is greater than the marginal cost, but the binding collateral constraint limits the house buyer's ability to borrow. The initial house price is $P_0 = 55.569$, and the mortgage loan contract price is $\pi = 26.431$, resulting in a Loan-to-Value ratio of 47.6%. The mortgage loan contract's face value is $A = 52.779$, so the interest rate of the mortgage loan is $r^{MORT} \equiv A/\pi - 1 = 0.997$. The house price at state U is $P_U = 72$, and we have $A\phi < \min[e_U^P, P_U\phi]$, so the house buyer is optimal and able to repay the mortgage loan in full. The house price at state D is $P_D = 45$, which is smaller than the mortgage loan repayment. So the house buyer chooses to default.

Table 2: Endogenous variables, two households two period binomial model

$P_0 = 55.569$	$c_0^R = 15.751$	$\pi = 26.431$	$\bar{U}^R = 5.557$
$P_U = 72$	$c_U^R = 31.502$	$\phi = 0.197$	$\bar{U}^P = -0.723$
$P_D = 45$	$c_D^R = 23.214$	$A = 52.779$	
$q_0 = 0.197$	$c_0^P = 4.249$	$r^{MORT} = 0.997$	
$q_U = 0.015$	$c_U^P = 8.498$	$LTV = 0.476$	
$q_D = 0.071$	$c_D^P = 1.786$		

The shock occurs to P's endowment at state D. The direct effect is, P has lower consumption at state D than in state U. Moreover, the low endowment leads to a low house price due to both income effect and substitution effect, as well as the endogenous default, which decrease R's income further at state D.

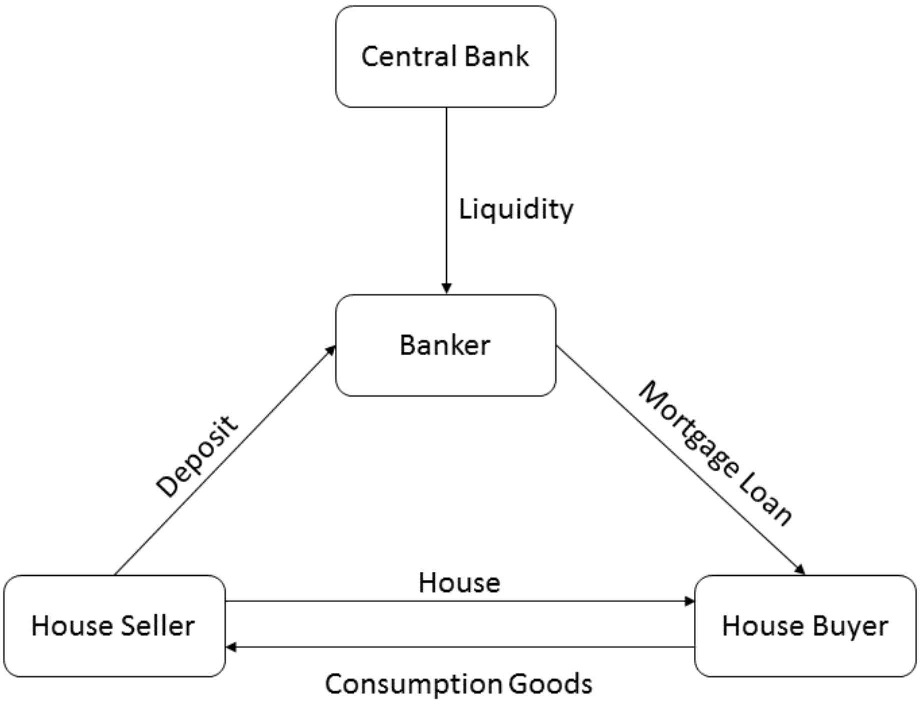
3 Simple mortgage model with a banker

The section above gives a simple model of mortgage loan contract directly between the house seller and buyer. However, non-professional individuals usually do not have the ability to create complex contracts, neither can they deal with seizing collateral easily. Moreover, endogenous contracts sometimes may fail to be traded between households, as shown in the following section. Assume now that the house sellers and buyers do not create mortgage loan contracts directly, but there is a banker in the economy, who can design and implement financial contracts at no cost. This model accommodates the one in Goodhart, Tsomocos, and Vardoulakis (2010), by removing the heterogeneity in the banks, and endogenizing the collateral constraint for the mortgage.

3.1 The economy

The binomial structure and the households' endowment are the same as in the previous model. The difference is, the households cannot create mortgage loan contract directly, and the banker offers the mortgage loan, as well as a deposit contract. The banker derives utility from consuming the consumption goods, and only cares about the consumption at $t = 1$. Figure 4 summarises the economy.

Figure 4: The Economy



The mortgage loan contract is similar as before, and the only difference is that now the contract is between the house buyer and the banker. The banker provides a safe deposit contract as well, promising a risk-free interest rate of r. Assume that the banker faces a cash-in-advance constraint. Within each period, she can borrow from the central bank for liquidity at the beginning of the state, and is obliged to repay at the end of the state, with an exogenous interest rate ($r^M \equiv \{r_0^M, r_U^M, r_D^M\}$). This gives the central bank some power to change the equilibrium. This model can only partially reflect the situation in the mortgage market. As mentioned in Gorton and Metrick (2012), the banks do not underwrite the mortgage loan itself, but outsource this to a direct lender "with a specialization of underwriting loans to be held for only a short time before being sold to banks". Moreover, our model leaves out securitization, which might be the most important character of the crisis. Nevertheless, we can think of the banker in our model serving all the roles of financial intermediation, without complicating the model too much.

At the initial state ($t = 0$), all the households and the banker enter the economy. The banker first borrows from the central bank for liquidity. The house buyer then purchases the houses from the seller, and borrows from the banker at the same time, using the mortgage loan contract. At the end of the period, the house seller make deposits to the banker, and the banker repays the central bank. At $t = 1$, if a good state (U) occurs, the buyer's new endowment is high, and the realized house price is high enough such that it is optimal for buyer to fully repay the mortgage loans. The banker borrows from the central bank first, repays the deposit to the house seller, and then get repaid from the house buyer. The buyer may purchase some more houses from the seller. The banker needs to repay the central bank at the end of the period. If a bad state (D) occurs, the house buyer defaults on the mortgage loan, either because the realised house price is so low that to default is optimal, or the borrower's endowment is too low. The banker borrows from the central bank first, repays the deposit to the house seller, and sells the houses in the market together with the house seller. The banker needs to repay the central bank at the end of the period.

3.2 The optimization problem

3.2.1 The house seller

The house seller's optimization problem is similar to the one in the previous model. The difference is that, instead of lending to the house buyer directly through a mortgage loan, the house seller make deposits to the banker. Endowed with some houses (E) at the beginning and the consumption goods at each state ($e^R \equiv \{e_0^R, e_U^R, e_D^R\}$), the house seller chooses her consumption ($c^R \equiv \{c_0^R, c_U^R, c_D^R\}$), the quantity of houses to sell ($q^R \equiv \{q_0^R, q_U^R, q_D^R\}$), and the amount to deposit (Dep^R) to maximize her expected utility,

$$\begin{aligned}\bar{U}^R =& U(c_0^R, E - q_0^R) + \beta^R\theta^R \left(U(c_U^R, E - q_0^R - q_U^R) + S(E - q_0^R - q_U^R)\right) \\ &+ \beta^R(1-\theta^R)\left(U(c_D^R, E - q_0^R + \phi^R - q_D^R) + S(E - q_0^R + \phi^R - q_D^R)\right)\end{aligned} \tag{3.1}$$

subject to the budget set $\mathcal{B}_2^R(P, r)$ defined by inequalities (3.2)~(3.4). At the initial state, her consumption plus the deposit should not exceed her endowment plus the income from selling the house,

$$c_0^R + Dep^R \leqslant e_0^R + P_0 q_0^R \tag{3.2}$$

If state U occurs, her consumption should not exceed the sum of her endowment, income from selling some more houses, and the deposit repayment,

$$c_U^R \leqslant e_U^R + P_U q_U^R + Dep^R(1 + r) \tag{3.3}$$

If state D occurs, her consumption should not exceed the sum of her endowment, income from selling some more houses, and the deposit repayment,

$$c_D^R \leqslant e_D^R + P_D q_D^R + Dep^R(1 + r) \tag{3.4}$$

Notice that if $q_D^R < 0$, the house seller would buy back some houses at state D.

3.2.2 The house buyer

The house buyer's optimization problem is the same as in the previous model. The only difference is that, the house buyer now borrows from the banker, instead of the house seller. The equations are the same as before. Denote her budget set as $\mathcal{B}_2^P(P, \pi)$, which is defined by inequalities (2.6)∼(2.9).

3.2.3 The banker

The banker derives utility from consuming the consumption goods only in the last period. Endowed with some initial capital (e^B) in terms of the consumption goods at $t = 0$, the banker chooses the borrowing from the central bank ($M \equiv \{M_0, M_U, M_D\}$), the deposit to absorb (Dep^B), the numbers of mortgage loan contract to issue (ϕ^B), and the mortgage loan's face value (A^B) to maximize her expected utility,

$$\bar{U}^B = \theta U(Profit_U) + (1-\theta)U(Profit_D) \tag{3.5}$$

subject to the budget set $\mathcal{B}_2^B(P, r, \pi, r^M)$ defined by inequalities (3.6)∼(3.9). At the initial state, the total mortgage loan issued should not exceed the initial capital plus the liquidity borrowed from the central bank,

$$\pi\phi \leqslant e^B + M_0 \tag{3.6}$$

the banker then absorbs deposit, and repays the central bank at the end of the initial state,

$$M_0(1 + r_0^M) \leqslant Dep^B \tag{3.7}$$

If the good state (U) occurs, the banker first borrows from the central bank to repay the deposit,

$$Dep(1 + r) \leqslant M_U \tag{3.8}$$

and the profit at this state is the mortgage loan repayment minus the repayment to the central bank,

$$Profit_U = A\phi^B - M_U(1 + r_U^M)$$

If the bad state (D) occurs, the banker still borrows from the central bank to repay the deposit first,

$$Dep(1 + r) \leqslant M_D \tag{3.9}$$

and the profit at this state is the income from selling the collateral minus the repayment to the central bank,

$$Profit_U = P_D\phi^B - M_D(1 + r_D^M)$$

3.3 The equilibrium

In equilibrium the house markets, the mortgage loan market, and the deposit market should all clear, namely the supply of the houses equals the demand for the houses at each state,

$$q_0^R = q_0^P \equiv q_0 \tag{3.10}$$
$$q_U^R = q_U^P \equiv q_U \tag{3.11}$$
$$q_D^R + \phi = q_D^P \tag{3.12}$$

the mortgage loan contract's supply equals demand,

$$\phi^B = \phi^P \equiv \phi \tag{3.13}$$

and the deposit supply equals demand,

$$Dep^R = Dep^B \equiv Dep \tag{3.14}$$

the banker and the house buyer agree on the mortgage loan's face value,

$$A^B = A^P \equiv A \tag{3.15}$$

and that the banker and the house buyer agree on the change in contract price with respect to face value around the equilibrium, for the same reason as mentioned in Section 2.3,

$$\frac{\partial \pi}{\partial A^B} = \frac{\partial \pi}{\partial A^P} \tag{3.16}$$

Definition 2. *The Collateral Equilibrium for the economy*

$$E_2 = \left\{ (\bar{U}^h, e^h, E)_{h \in \{R,P,B\}}, r^M \right\}$$

is a vector consisting of house prices, mortgage loan contract price, deposit interest rate, consumptions, house trading, mortgage loan face value and traded volume, and deposit volume,

$$\Big((P, \pi, r), (c^h, q^h)_{h \in \{R,P\}}, A, \phi, Dep \Big)$$

if and only if

1. $(c^R, q^R, Dep^R) \in \mathcal{B}_2^R(P, r)$
   $(\tilde{c}, \tilde{q}, \tilde{Dep}) \in \mathcal{B}_2^R(P, \pi) \Rightarrow \bar{U}^R(\tilde{c}, \tilde{q}) \leqslant \bar{U}^R(c^R, q^R)$

2. $(c^P, q^P, \phi^P, A^P) \in \mathcal{B}_2^P(P, \pi)$
   $(\tilde{c}, \tilde{q}, \tilde{\phi}, \tilde{A}) \in \mathcal{B}_2^P(P, r) \Rightarrow \bar{U}^P(\tilde{c}, \tilde{q}) \leqslant \bar{U}^P(c^P, q^P)$

3. $(M, Dep^B, \phi^B, A^B) \in \mathcal{B}_2^B(P, \pi, r)$
   $(\tilde{M}, \tilde{Dep}, \tilde{\phi}, \tilde{A}) \in \mathcal{B}_2^B(P, \pi, r) \Rightarrow \bar{U}^B(\tilde{M}, \tilde{Dep}, \tilde{\phi}, \tilde{A}) \leqslant \bar{U}^B(M, Dep^B, \phi^B, A^B)$

4. *Market clearing conditions (3.10)∼(3.16) hold.*

5. *In equilibrium* $\min\left[\frac{e_D^P}{\phi}, P_D\right] < A < \min\left[\frac{e_U^P}{\phi}, P_U\right]$

3.4 Numerical Example

Choose the exogenous variables and utility functions the same as in the previous section. The difference is that, now we have a banker, with some initial endowment, and borrows from the central bank for liquidity at exogenous interest rates. The exogenous variables are given in Table 3, and the banker's utility function is,

$$\bar{U}^B = \theta^B \ln(Profit_U) + (1-\theta^B)\ln(Profit_D)$$

Table 3: Exogenous variables, two households two period binomial model with a banker

$E = 1$	$\theta = 0.99$	$\tau = 0.8$	$\beta^R = 1$	$\beta^P = 0.8$	$r_0^M = 0$	$r_U^M = 0$	$r_D^M = 0$
$e_0^R = 10$	$e_U^R = 20$	$e_D^R = 20$	$e_0^P = 10$	$e_U^P = 20$	$e_D^P = 5$	$e^B = 1$	

The endogenous variables are given in Table 4. In equilibrium, the house buyer's collateral constraint binds. The initial house price is $P_0 = 58.689$, and the mortgage loan contract price is $\pi = 29.171$, resulting in a Loan-to-Value ratio of 49.7%. The house price at state U is $P_U = 68.607$, and we have $A\phi < \min[e_U^P, P_U\phi]$, so the house buyer is optimal and able to repay the mortgage loan in full. The house price at state D is $P_D = 44.455$, which is smaller than the mortgage loan repayment. So the house buyer chooses to default. $q_D^R < 0$ means the house seller buys back some houses at the state D, since the house price is low, while her endowment is high enough.

Table 4: Endogenous variables, two households two period binomial model with a banker

$P_0 = 58.689$	$c_0^R = 16.636$	$H_0^R = 0.809$	$\pi = 29.171$	$M_0 = 4.570$	$\bar{U}^R = 5.560$
$P_U = 68.607$	$c_U^R = 29.890$	$H_U^R = 0.784$	$\phi = 0.191$	$M_U = 8.186$	$\bar{U}^P = -0.7323$
$P_D = 44.455$	$c_D^R = 22.912$	$H_D^R = 0.928$	$A = 52.742$	$M_D = 8.186$	$\bar{U}^B = 0.616$
$q_0 = 0.191$	$c_0^P = 4.364$	$H_0^P = 0.191$	$r^{MORT} = 0.808$	$D = 4.570$	
$q_U = 0.248$	$c_U^P = 8.225$	$H_U^P = 0.216$	$LTV = 49.7\%$	$r = 0.791$	
$q_D^R = -0.119$	$c_D^P = 1.786$	$H_D^P = 0.072$			

We would like to see the effect of the banker. If the banker may choose to stay away from the economy, her utility is simply $\theta^B \ln 1 + (1-\theta)\ln 1 = 0$. Her expected utility from serving as financial intermediary is $0.616 > 0$, so it's optimal for her to stay in the economy. To see the effect on the house traders, we need a benchmark economy where the households can only trade house with the consumption goods directly, without being able to create any contract. The calculations are given in Appendix B.1. With the same choice of exogenous variables, the endogenous variables in equilibrium are given in Table 5.

Table 5: Endogenous variables, benchmark model

$P_0 = 56.962$	$P_U = 72$	$P_D = 45$	$q_0 = 0.109$	$q_U = 0.140$	$q_D = 0.033$
$c_0^R = 16.206$	$c_U^R = 30.056$	$c_D^R = 21.463$	$c_0^P = 3.794$	$c_U^P = 9.944$	$c_D^P = 3.537$
$\bar{U}^R = 5.558$	$\bar{U}^R = -1.067$				

Compared with the benchmark case, the existence of the banker makes both the house sellers and the buyers better off. Thus, Pareto improvement is achieved through the banker in this case. One more thing to notice that, the expected utility of the house seller in Section 2.4 is smaller than that in the benchmark case. This means the house sellers are better off if there is no financial contract at all, compared with creating mortgage loan contract directly between the house sellers and buyers. Thus the equilibrium in Section 2.4 doest not exist, and no financial contract will be traded without the help of the banker.

Proposition 3. *The households may not always be able contract directly, either because of lack of ability, or the fact that the endogenous contract cannot achieve Pareto improvement, thus is not traded in equilibrium. The financial intermediary can solve the problem by offering better risk-sharing, and making the market more complete.*

The monetary authority (the central bank in this model) can change the equilibrium variables by changing the monetary policy. In this case, I assume that the monetary tool is the interest rate for the central bank lending. Table 6 shows how the endogenous variables and welfare are affected by an increase in the central bank interest rate. Generally, an increase in the interest rate means a decrease in the money supply. In this model however, the increase in one state can affect the central bank lending at all the states. This is because the banker have rational expectation, and that the lending at different time are related by the banker's activity. For example, an increase in r_0 decrease M_0, and thus decrease the banker's ability to issue mortgage loan. Demand for the houses are lower, and so is the house price. The house seller has less income, thus deposits less. The banker then needs to borrow less from the central bank in the later period.

Table 6: Central bank interest rate and endogenous variables

	P_0	P_U	P_D	q_0	q_U	q_D	π	A	r^{MORT}	LTV	Dep	r	M_0	M_U	M_D	$\bar{U}^R$	$\bar{U}^P$	$\bar{U}^B$
r_0	−	−	−	−	+	−	−	−	+	−	−	+	−	−	−	−	+	+
r_U	−	−	−	−	+	−	−	−	+	−	−	−	−	−	−	−	+	+
r_D	+	−	−	−	+	−	−	−	−	−	−	−	−	−	−	−	+	+

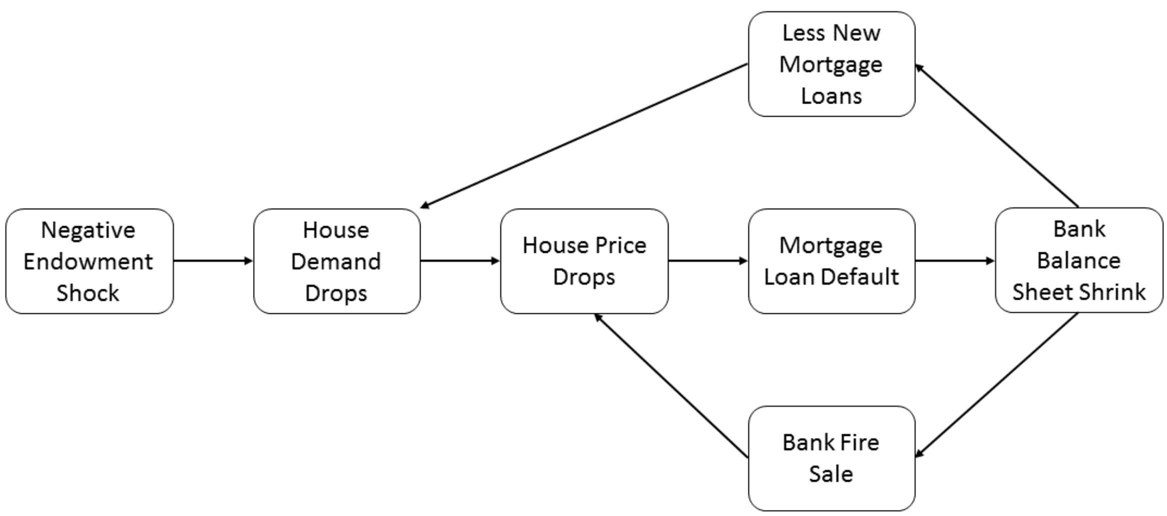
Notice that the central bank cannot move all the households' utility in the same direction at the same time. Thus the central bank can only redistribute welfare, but not make Pareto improvement by monetary policy only.

4 Credit cycle

The model above shows how the financial intermediary (the banker) makes Pareto improvement in a basic binomial framework. In this section we study how the banker affects the economy in a longer term.

The following model is an extension of the previous one, with three-period binomial structure, overlapping generation households, and a long-living banker. An endowment shock occurs at state D to the house buyers, decreasing the demand for houses. Thus the house prices decrease, which triggers mortgage loan default. The banker's balance sheet shrinks accordingly, and less credit is available for new mortgage loans. This further decreases the demand for houses and the prices, which forms a credit cycle. Also, the banker needs to resell the seized collateral at the housing market, where the demand is already low. This is similar to the Shleifer and Vishny (1992) style fire sale, which further decrease the house price. The two cycles reinforce each other, persisting and amplifying the effect of negative endowment shock. If things go extreme, there might be bank failure. Brunnermeier and Pedersen (2009) provides a similar twin-spiral structure, and links the asset's market liquidity with the traders' funding liquidity. Their analysis focus on the speculator's position in financial assets, where initial loss leads to funding problems for speculators, who reduce positions, making prices move further away from fundamentals. Higher margins and loss on existing positions makes the funding problems for the speculators worse. The banker in our model plays a similar role as the speculators in their model. Figure 5 illustrates how the credit cycle and the price cycle reinforce each other, amplifying the initial endowment shock.

Figure 5: Credit Cycle



The endowment shock to the house buyers have asymmetric effects on the equilibrium variables. Higher endowment leads to higher house price due to income and substitution effects, but lower endowment leads to default in the mortgage loan and bank fire sale in addition. The frictions created by the need for collateral creates asymmetry between the boom and bust time of the economy.

The house provides both fundamental value and collateral value to the house buyers, as we mentioned in Section 2. When the house price drops in the bad states, the value of loans backed by one unit of house also drops. This decrease the collateral value of the house as well. The dual role of durable goods as fundamental use and collateral is captured by Kiyotaki and Moore (1997). The dynamic interaction between credit limits and asset prices in their model resembles the transmission mechanism by which the effect of shocks persist, amplify and spill over to other sectors. Our model adopts this idea with non-production factors, namely the houses.

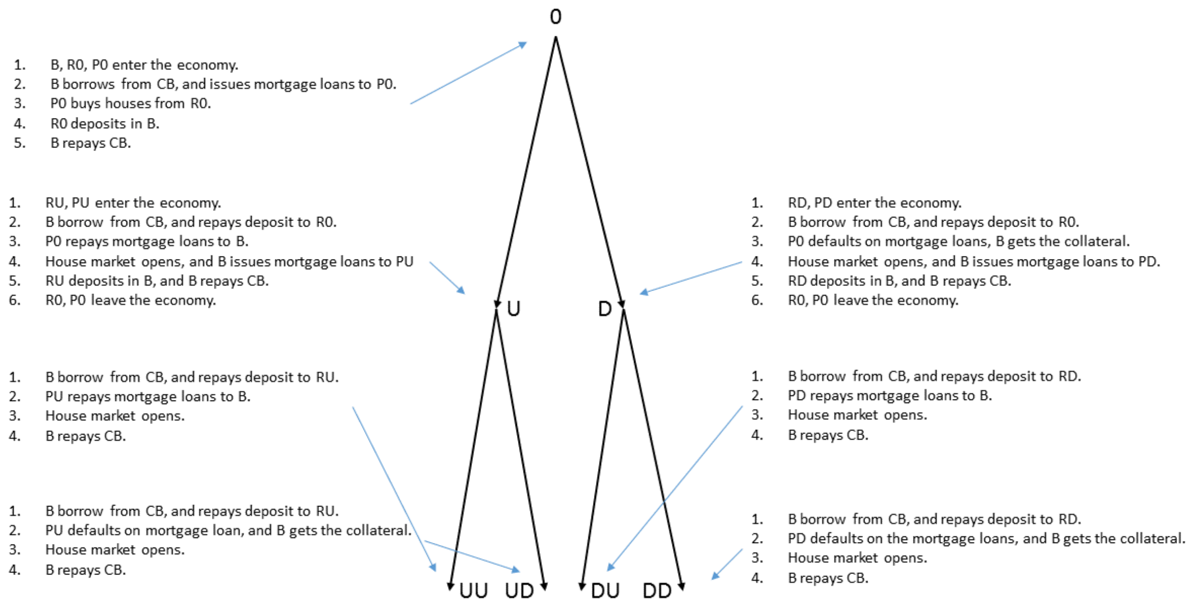
Our model predicts that the leverage of the mortgage loan contract co-moves with the house price. This is because the shrinkage of the banker's balance sheet limits the credit for new mortgage loans thus the demand for new houses, while the fire sale keeps the supply of houses high. Thus the low house price accompanies low credit, and a risk-averse banker would demand higher proportion of down-payment, thus low leverage as well. Geanakoplos (2010) describes such situation as leverage cycle, and suggests that the in the bad times, the belief disagreement among the households increases, while the overall optimism decreases due to the removal of the most-optimistic, highly leveraged agents after default. Our model discusses the change in optimism in the next section, and shows that if the banker becomes more pessimistic, the effect of the initial shock will be exacerbated. Adrian and Shin (2010) documents a strong positive relationship between changes in leverage and changes in balance sheet size, and implies that financial intermediaries, especially security brokers and dealers, adjust their balance sheets actively, leading to a procylical leverage. Our model suggests that, when the balance sheet of the banker is actively marked to market, the financial contract created by the banker has a pro-cyclical leverage ratio.

4.1 The economy

Consider a three-period ($t = 0, 1, 2$) binomial economy with two types of households (house sellers and house buyers) and a banker. The basic settings are similar to those in the previous section, with some extensions. The house seller $R0$ and house buyer $P0$ enter the economy at the beginning of $t = 0$, and leave the economy at the end of $t = 1$. The house seller RU and house buyer PU enter the economy at the beginning of state U, $t = 1$, and leave the economy at the end of $t = 2$. The house seller RD and house buyer PD enter the economy at the beginning of state D, $t = 1$, and leave the economy at the end of $t = 2$. In fact, RU and RD are the same household, only entering the economy at different states of the same period. So are PU and PD. The banker lives throughout the time. Long story short, this is an overlapping generation model with a long-living banker.

Figure 6 shows the time structure of the economy. At the beginning of the initial state, the banker (B), the house seller ($R0$), and the house buyer ($P0$) enter the economy. B first borrows from the central bank, and issues mortgage loans to $P0$. $R0$ and $P0$ then trade houses with

Figure 6: Time Structure



consumption goods, and $R0$ makes deposit to the banker. B repays fully to the central bank, and the households consume. If state U occurs at $t = 1$, $P0$'s endowment is high. At the beginning of state U, RU and PU enters the economy. B borrows from the central bank, and repays deposit to $R0$. Then $P0$ repays the mortgage loan, and B issues new mortgage loan to PU. Households trade houses with consumption goods, and RU deposit in B. By the end of the state, household consumes and the banker repays the central bank. $R0$ and $P0$ leave the economy. If state UU occurs after U, PU's endowment is high. B borrows from the central bank, and repays deposit to RU. Then PU repays the mortgage loan, and households trade and consume. B repays the central bank by the end of the state, and realizes the profit. If state UD occurs after U, PU's endowment is low. B borrows from the central bank, and repays deposit to RU. PU's endowment is so low such that either she cannot afford to repay the mortgage loan, or the equilibrium house price is lower than the mortgage repayment. So PU defaults on the mortgage loan, B gets the collateral, and resells. Then households and the banker trade and consume. B repays the central bank by the end of the state, and realizes the profit. The time structure is similar if state D occurs at $t = 1$. The only difference is, $P0$ defaults at D, the banker seizes the collateral, and resells either immediately or in the next period. Notice that, when households leave the economy, they take away the houses owned at the end of the period, and the houses provide some salvage value to the households.

4.2 The optimization problem

The optimization problems for $R0$, RU and RD, are exactly the same as those for the house seller R in the previous model in Section 3.2.1, while the optimization problems for $P0$, PU and PD, are exactly the same as those for the house buyer P in the previous model in Section 3.2.2. The optimization problem for the banker is below.

The **banker** derives utility from consuming the consumption goods only in the last period. Endowed with some initial capital (e^B) in terms of the consumption goods in the initial state, the banker chooses the borrowing from the central bank ($M \equiv \{M_0, M_U, M_{UU}, M_{UD}, M_D, M_{DU}, M_{DD}\}$), the deposit to absorb ($Dep \equiv \{Dep_0^B, Dep_U^B, Dep_D^B\}$), the numbers of mortgage loan contract to issue ($\phi \equiv \{\phi_0^B, \phi_U^B, \phi_D^B\}$), the mortgage loan's face value ($A \equiv \{A_0^B, A_U^B, A_D^B\}$), and the amount of houses to sell at state D (q_D^B) to maximize her expected utility,

$$\begin{aligned}\bar{U}^B = &(\theta^B)^2 U(Profit_{UU}) + \theta^B(1-\theta^B)U(Profit_{UD}) \\ &+ \theta^B(1-\theta^B)U(Profit_{DU}) + (1-\theta^B)^2 U(Profit_{DD})\end{aligned} \tag{4.1}$$

subject to the budget set $\mathcal{B}_3^B(P, r, \pi, r^M)$ defined by inequalities (4.2)∼(4.12), where
$P \equiv \{P_0, P_U, P_{UU}, P_{UD}, P_D, P_{DU}, P_{DD}\}$ is the vector of house price,
$r \equiv \{r_0, r_U, r_D\}$ is the vector of deposit interest rates,
$\pi \equiv \{\pi_0, \pi_U, \pi_D\}$ is the vector of mortgage loan prices,
$r^M \equiv \{r_0^M, r_U^M, r_{UU}^M, r_{UD}^M, r_D^M, r_{DU}^M, r_{DD}^M\}$ is the vector of central bank lending interest rate.

At the initial state, the total mortgage loan issued should not exceed the initial capital plus the liquidity borrowed from the central bank,

$$\pi_0 \phi_0^B \leqslant e_0^B + M_0 \tag{4.2}$$

then the banker should absorb enough deposit to repay the central bank,

$$M_0(1 + r_0^M) \leqslant Dep_0^B \tag{4.3}$$

At state U, the banker first borrows from the central bank to repay the deposit,

$$Dep_0^B(1 + r_0) \leqslant M_U \tag{4.4}$$

then the issuance of new mortgage loans plus the repayment to the central bank should not exceed the collected mortgage repayment plus the new deposit absorbed,

$$\pi_U \phi_U^B + M_U(1 + r_U^M) \leqslant A_0^B \phi_0^B + Dep_U^B \tag{4.5}$$

At state UU, the banker first borrows from the central bank to repay the deposit,

$$Dep_U^B(1 + r_U) \leqslant M_{UU} \tag{4.6}$$

and the realized profit is the collected mortgage repayment minus the repayment to the central bank,

$$Profit_{UU} = A_U^B \phi_U^B - M_{UU}(1 + r_{UU}^M)$$

At state UD, the banker first borrows from the central bank to repay the deposit,

$$Dep_U^B(1 + r_U) \leqslant M_{UD} \tag{4.7}$$

and the realized profit is the income from reselling the collateral, minus the repayment to the central bank,

$$Profit_{UD} = P_{UD}\phi_U^B - M_{UD}(1 + r_{UD}^M)$$

At state D, the banker first borrows from the central bank to repay the deposit,

$$Dep_0^B(1 + r_0) \leqslant M_D \tag{4.8}$$

then the issuance of new mortgage loans plus the repayment to the central bank should not exceed the resale of collateral plus the new deposit absorbed,

$$\pi_D\phi_D^B + M_D(1 + r_D^M) \leqslant P_D q_D^B + Dep_D^B \tag{4.9}$$

notice that the banker cannot sell more than the seized collateral,

$$q_D^B \leqslant \phi_0^B \tag{4.10}$$

At state DU, the banker first borrows from the central bank to repay the deposit,

$$Dep_D^B(1 + r_D) \leqslant M_{DU} \tag{4.11}$$

and the realized profit is the mortgage repayment, plus the resale of the collateral seize at state D, minus the repayment to the central bank,

$$Profit_{DU} = A_D^B\phi_D^B + P_{DU}(\phi_0^B - q_D^B) - M_{DU}(1 + r_{DU}^M)$$

At state DD, the banker first borrows from the central bank to repay the deposit,

$$Dep_D^B(1 + r_D) \leqslant M_{DD} \tag{4.12}$$

and the realized profit is the resale of the collateral, minus the repayment to the central bank,

$$Profit_{DD} = P_{DD}(\phi_D^B + \phi_0^B - q_D^B) - M_{DD}(1 + r_{DD}^M)$$

4.3 The equilibrium

In equilibrium, the markets for the houses, mortgage loans, and deposit should all clear. More specifically, the supply and demand of houses should equal in equilibrium at all states,

$$q_0^{R0} = q_0^{P0} \tag{4.13}$$

$$q_U^{R0} + q_U^{RU} = q_U^{P0} + q_U^{PU} \tag{4.14}$$

$$q_{UU}^{RU} = q_{UU}^{PU} \tag{4.15}$$

$$q_{UD}^{RU} + \phi_U^B = q_{UD}^{PU} \tag{4.16}$$

$$q_D^{R0} + q_D^{RD} + q_D^B = q_D^{P0} + q_D^{PD} \tag{4.17}$$

$$q_{DU}^{RD} + \phi_0^B - q_D^B = q_{DU}^{PD} \tag{4.18}$$

$$q_{DD}^{RD} + \phi_0^B - q_D^B + \phi_D^B = q_{UD}^{PD} \tag{4.19}$$

The mortgage loan supply from the banker equals the demand from the house buyers,

$$\phi_0^B = \phi_0^{P0} \equiv \phi_0 \tag{4.20}$$

$$\phi_U^B = \phi_U^{PU} \equiv \phi_U \tag{4.21}$$

$$\phi_D^B = \phi_D^{PD} \equiv \phi_D \tag{4.22}$$

The face value of the mortgage loan contract chosen by the banker and the house buyers should be the same,

$$A_0^B = A_0^{P0} \equiv A_0 \tag{4.23}$$

$$A_U^B = A_U^{PU} \equiv A_U \tag{4.24}$$

$$A_D^B = A_D^{PD} \equiv A_D \tag{4.25}$$

The banker and the house buyers agree on the change in mortgage loan contract price with respect to change in face value around the equilibrium, for the same reason as mentioned in Section 2.3,

$$\frac{\partial \pi}{\partial A_0^B} = \frac{\partial \pi}{\partial A_0^{P0}} \tag{4.26}$$

$$\frac{\partial \pi}{\partial A_U^B} = \frac{\partial \pi}{\partial A_U^{PU}} \tag{4.27}$$

$$\frac{\partial \pi}{\partial A_D^B} = \frac{\partial \pi}{\partial A_D^{PD}} \tag{4.28}$$

The demand for deposit of the banker should equal the supply of deposit of the house sellers,

$$Dep_0^B = Dep_0^{R0} \equiv Dep_0 \tag{4.29}$$

$$Dep_U^B = Dep_U^{RU} \equiv Dep_U \tag{4.30}$$

$$Dep_D^B = Dep_D^{RD} \equiv Dep_D \tag{4.31}$$

Definition 3. *The Collateral Equilibrium for the economy*

$$E_3 = \left\{ (\bar{U}^h, e^h)_{h \in H}, r^M \right\}$$

is a vector consisting of house prices, mortgage loan contract prices, deposit interest rates, consumptions, house trading, mortgage loan face value and traded volumes, and deposit volumes,

$$\left((P, \pi, r), (c^h, q^h)_{h \in H \setminus B}, q_D^B, A, \phi, Dep \right)$$

if and only if

1. $(c^{Rs}, q^{Rs}, Dep^{Rs}) \in \mathcal{B}_3^{Rs}(P, r), s \in \{0, U, D\}$
   $(\tilde{c}, \tilde{q}, \tilde{Dep}) \in \mathcal{B}_3^{Rs}(P, \pi) \Rightarrow \bar{U}^{Rs}(\tilde{c}, \tilde{q}) \leqslant \bar{U}^{Rs}(c^{Rs}, q^{Rs})$

2. $(c^{Ps}, q^{Ps}, \phi^{Ps}, A^{Ps}) \in \mathcal{B}_3^{Ps}(P, \pi), s \in \{0, U, D\}$
   $(\tilde{c}, \tilde{q}, \tilde{\phi}, \tilde{A}) \in \mathcal{B}_2^{Ps}(P, r) \Rightarrow \bar{U}^{Ps}(\tilde{c}, \tilde{q}) \leqslant \bar{U}^{Ps}(c^{Ps}, q^{Ps})$

3. $(M, Dep^B, \phi^B, A^B) \in \mathcal{B}_3^B(P, \pi, r)$
   $(\tilde{M}, \tilde{Dep}, \tilde{\phi}, \tilde{A}) \in \mathcal{B}_3^B(P, \pi, r) \Rightarrow \bar{U}^B(\tilde{M}, \tilde{Dep}, \tilde{\phi}, \tilde{A}) \leqslant \bar{U}^B(M, Dep^B, \phi^B, A^B)$

4. *Market clearing conditions (4.13)∼(4.31) hold.*

5. *In equilibrium* $\min\left[\frac{e_{sD}^{Ps}}{\phi_s}, P_{sD}\right] < A_s < \min\left[\frac{e_{sU}^{Ps}}{\phi_s}, P_{sU}\right], s \in \{0, U, D\}$.

where
$H \equiv \{R0, RU, RD, P0, PU, PD, B\}$ *is the set of all the agents in the economy,*
$e^{Rs} \equiv \{E_s, e_s^{Rs}, e_{sU}^{Rs}, e_{sD}^{Rs}\}, s \in \{0, U, D\}$ *is the vector of the endowment of the house sellers,*
$e^{Ps} \equiv \{e_s^{Ps}, e_{sU}^{Ps}, e_{sD}^{Ps}\}, s \in \{0, U, D\}$ *is the vector of the endowment of the house buyers,*
e^B *is the bankers endowment,*
$r^M \equiv \{r_0^M, r_U^M, r_{UU}^M, r_{UD}^M, r_D^M, r_{DU}^M, r_{DD}^M\}$ *is the vector of central bank lending interest rates,*
$P \equiv \{P_0, P_U, P_{UU}, P_{UD}, P_D, P_{DU}, P_{DD}\}$ *is the vector of house prices,*
$\pi \equiv \{\pi_0, \pi_U, \pi_D\}$ *is the vector of mortgage loan contract prices,*
$A \equiv \{A_0, A_U, A_D\}$ *is the vector of mortgage loan contract face values,*
$Dep \equiv \{Dep_0, Dep_U, Dep_D\}$ *is the vector of deposits,*
$r \equiv \{r_0, r_U, r_D\}$ *is the vector of deposit interest rates,*
$M \equiv \{M_0, M_U, M_{UU}, M_{UD}, M_D, M_{DU}, M_{DD}\}$ *is the vector of the banker's lending from the central bank.*

4.4 Numerical Example

The exogenous variables are given in Table 7. Each house seller has the same endowment in consumption goods across time, while each house buyer has relatively higher endowment in the good state and lower endowment in the bad state. As we assume, RU and RD are the same household entering the economy at different states of the same time period, thus RU and RD have the same endowment. So are PU and PD. Assume for now that all the households and the banker have the same belief on the states (same θ), they remain constant.

Table 7: Exogenous variables, multi-period model

$r_0^M = 0$	$r_U^M = 0$	$r_{UU}^M = 0$	$r_{UD}^M = 0$	$r_D^M = 0$	$r_{DU}^M = 0$	$r_{DD}^M = 0$
$e_0^B = 2$	$\theta^B = 0.95$	$\tau = 0.6$				
$E_0^{R0} = 1$	$e_0^{R0} = 10$	$e_U^{R0} = 10$	$e_D^{R0} = 10$	$\beta^{R0} = 1$	$\theta^{R0} = 0.95$	
	$e_0^{P0} = 10$	$e_U^{P0} = 20$	$e_D^{P0} = 10$	$\beta^{P0} = 0.8$	$\theta^{P0} = 0.95$	
$E_U^{RU} = 1$	$e_U^{RU} = 22$	$e_{UU}^{RU} = 22$	$e_{UD}^{RU} = 22$	$\beta^{RU} = 1$	$\theta^{RU} = 0.95$	
	$e_U^{PU} = 22$	$e_{UU}^{PU} = 44$	$e_{UD}^{PU} = 22$	$\beta^{PU} = 0.8$	$\theta^{PU} = 0.95$	
$E_D^{RD} = 1$	$e_D^{RD} = 22$	$e_{DU}^{RD} = 22$	$e_{DD}^{RD} = 22$	$\beta^{RD} = 1$	$\theta^{RD} = 0.95$	
	$e_D^{PD} = 22$	$e_{DU}^{PD} = 44$	$e_{DD}^{PD} = 22$	$\beta^{PD} = 0.8$	$\theta^{PD} = 0.95$	

Also, let all the instantaneous utility functions be logarithm, and the salvage value function be $S(\cdot) = \tau U(\cdot)$, thus we have the expected utility functions as below,

$$\begin{aligned}
\bar{U}^B &= (\theta^B)^2 \ln(Profit_{UU}) + \theta^B(1-\theta^B)\ln(Profit_{UD}) \\
&\quad + \theta^B(1-\theta^B)\ln(Profit_{DU}) + (1-\theta^B)^2 \ln(Profit_{DD}) \\
\bar{U}^{Rs} &= \ln(c_s^{Rs}) + \ln(E_s^{Rs} - q_s^{Rs}) + \beta^{Rs}\theta^{Rs}\left(\ln(c_{sU}^{Rs}) + (1+\tau)\ln(E_s^{Rs} - q_s^{Rs} - q_{sU}^{Rs})\right) \\
&\quad + \beta^{Rs}(1-\theta^{Rs})\left(\ln(c_{sD}^{Rs}) + (1+\tau)\ln(E_s^{Rs} - q_s^{Rs} - q_{sD}^{Rs})\right) \quad s \in \{0, U, D\} \\
\bar{U}^{Ps} &= \ln(c_s^{Ps}) + \ln(q_s^{Ps}) + \beta^{Ps}\theta^{Ps}\left(\ln(c_{sU}^{Ps}) + (1+\tau)\ln(q_s^{Ps} + q_{sU}^{Ps})\right) \\
&\quad + \beta^{Ps}(1-\theta^{Ps})\left(\ln(c_{sD}^{Ps}) + (1+\tau)\ln(q_s^{Ps} - \phi_s^{Ps} + q_{sD}^{Ps})\right) \quad s \in \{0, U, D\}
\end{aligned}$$

With the given choice of exogenous variables and utility functions, the equilibrium has the following characters: 1) the banker is solvent in all the states; 2) all the budget constraint binds; 3) all the collateral constraints for the house buyers binds; 4) the house buyers default on the mortgage loans at the bad states (D, UD, DD), and repays fully in the good states (U, UU, DU); 5) the banker resells all the collateral as soon as possible. These properties are satisfied when all the households and the banker optimize over their choice variables.

Table 8 presents the endogenous variables given the choice of the parameters and utility functions. The house prices increase in the good states, and decrease in the bad states. The drop in the house price leads to default accordingly. As the model is simplified and assumes away the necessity for the bank to hold capital buffer or extra liquidity, the borrowing from the central bank is determined by the deposit absorbed at the previous state. Thus $M_U = M_D$, $M_{UU} = M_{UD}$ and $M_{DU} = M_{DD}$. As the model has only three periods, boundary conditions affect variables at different time significantly. However, we can compare variables at different states but in the same period, sharing the same previous state, namely we can compare the variables in state U and D. As we set the probability of state D to be only 5%, we can think of the state as a rare event, and the low endowment to be a negative shock.

The negative endowment shock decreases the mortgage loan contract's price, as well as the face value. ($\pi_D < \pi_U$ and $A_D < A_U$). The Loan-to-Value ratio (LTV) is lower, and the required rate of return is higher at state D compared with those at state U. This is because, the default

Table 8: Endogenous variables, multi-period model

	0	U	UU	UD	D	DU	DD
M	7.082	13.015	18.481	18.481	13.015	19.859	19.859
π	40.477	43.125			33.014		
ϕ	0.224	0.346			0.364		
A	75.305	76.169			66.618		
$\pi \cdot \phi$	9.082	14.901			12.002		
r^{MORT}	0.860	0.766			1.018		
LTV	0.644	0.570			0.499		
Dep	7.082	11.020			10.171		
r	0.838	0.677			0.952		
$r^{MORT} - r$	0.023	0.089			0.065		
P	62.862	75.715	79.379	57.947	66.165	81.130	56.849
c^{R0}	17.023	31.439			28.590		
q^{R0}	0.224	0.111			0.084		
H^{R0}	0.776	0.664			0.691		
c^{P0}	4.977	7.728			3.846		
q^{P0}	0.224	-0.061			0.093		
H^{P0}	0.224	0.163			0.093		
c^{RU}		24.094	40.812	33.997			
q^{RU}		0.173	0.004	-0.112			
H^{RU}		0.827	0.823	0.939			
c^{PU}		10.739	17.350	8.462			
q^{PU}		0.346	0.004	0.234			
H^{PU}		0.346	0.350	0.234			
c^{RD}					21.616	42.688	34.730
q^{RD}					0.148	0.010	-0.125
H^{RD}					0.852	0.842	0.977
c^{PD}					9.948	18.952	8.462
q^{PD}					0.364	0.010	0.238
H^{PD}					0.364	0.374	0.238

in the mortgage loans at state D reduce the size of the banker's balance sheet, thus less credit is available for new mortgage loans. The spread between the mortgage interest rate and the deposit interest rate, which measures the profitability of the banker, is lower at state D. The stressed financial condition makes the banker require stricter terms for the new mortgage loans. The total credit for mortgage loans ($\pi\phi$) at state D is also lower than that of state U. Although the house price is lower at D, the tougher credit conditions decreases the demand for the houses from the new house buyers, further decreasing the house price. This is the credit cycle. In equilibrium, it is optimal for the banker to resell the collateral as soon as possible according to our calculation. This fire sale decreases the house price at the bad state as well. New house seller's income is lower in state D than that in state U, due to the lower house price, lower house demand, and the competition from the banker reselling the collateral. Thus, supply of deposit is less, and the interest rate is consequently higher.

For each of the house buyers and sellers, the consumption of the goods is higher in the good states, and lower in the bad states. Notice that $q_U^{P0} < 0$, $q_{UD}^{RU} < 0$, and $q_{DD}^{RD} < 0$. $q_U^{P0} < 0$ implies that, at state U, the house price is so high that the initial house buyer $P0$ would sell some of her house bought previously, after repaying the mortgage loans. The houses transfer some wealth across states, and may serve as means of speculation. $q_{UD}^{RU} < 0$ and $q_{DD}^{RD} < 0$ imply that, the house price at the bad states in the final period is so low, that the second generation house sellers RU and RD find it optimal to buy some houses back for their own living.

In summary, when a negative shock occurs to the house buyer's endowment, the house price decreases, which triggers default on the mortgage loans. The banker's balance sheet shrinks, and offers less credit for new mortgage loans, with stricter conditions. The collateral value of the house decreases together with the fundamental value, thus further decrease the house price. Moreover, the banker resells the collateral in the market, creating a situation similar situation to a fire sale. All these factors contribute to a credit cycle.

5 Belief formation and Minsky's financial instability hypothesis

Minsky (1977) states that, over a period in which the economy does well, optimism changes, and agents tend to invest more in the riskier asset. The tendency to transform doing well into a speculative investment boom is the basic instability in the economy. In this section we study such a behaviour of the banker, and the consequences for the equilibrium. The economy and optimization problems are the same as in Section 4. The only difference is that, the true state of nature, which is the probability of a good state to occur in the next period (θ), is unknown. The banker holds a prior belief on θ, and learn about the true value following the Bayes' rule, by observing the realization of states. Assume that the households do not learn from observing the states, since they live for only two periods, or simply because they lack such an ability.

5.1 Beliefs

We model the Bayesian learning of beliefs in the same way as Bhattacharya, Goodhart, Tsomocos, and Vardoulakis (2015), which follows Cogley and Sargent (2008). Assume that the true state of the nature θ, can only take two values, θ_1 or θ_2, with $0 < \theta_2 < \theta_1 < 1$. At a given state, the subsequent state can be either good (U), with probability θ, or bad (D), with probability $1 - \theta$. Assume that θ does not change throughout time.

The households and the banker have rational expectation in the sense that they are aware of what is going to happen in each of the state, i.e. they know the prices, the interest rate and the consumption level if a certain state occurs. The households do not learn about θ from the realization of states, and assume that their beliefs are constant and homogeneous. The banker knows about the two values θ_1 and θ_2, and have her subjective prior belief ρ on θ initially. This means,

$$\theta_0^B = \rho\theta_1 + (1-\rho)\theta_2 \tag{5.1}$$

If state U occurs,

$$\theta_U^B = \theta_1 Prob(\theta = \theta_1|U) + \theta_2 Prob(\theta = \theta_2|U) = \frac{\theta_1^2\rho + \theta_2^2(1-\rho)}{\theta_1\rho + \theta_2(1-\rho)} \tag{5.2}$$

If state D occurs,

$$\theta_D^B = \theta_1 Prob(\theta = \theta_1|D) + \theta_2 Prob(\theta = \theta_2|D) = \frac{\theta_1(1-\theta_1)p + \theta_2(1-\theta_2)(1-p)}{(1-\theta_1)p + (1-\theta_2)(1-p)} \tag{5.3}$$

It is easy to show that $\theta_D^B < \theta_0^B < \theta_U^B$, which means the realization of a good state increases optimism, while the realization of a bad state decreases optimism. The calculations are in Appendix B.2. Using the same parameters as in Section 4.4, and the subjective probabilities given in Table 9 below, we can see how Bayesian learning of the banker affect the economy in equilibrium.

Table 9: Beliefs

$\theta_1 = 0.99$	$\theta_2 = 0.91$	$\rho = 0.5$	$\theta_0^B = 0.95$	$\theta_U^B = 0.952$	$\theta_D^B = 0.918$
$\theta^{R0} = 0.95$	$\theta^{P0} = 0.95$	$\theta^{RU} = 0.95$	$\theta^{PU} = 0.95$	$\theta^{RD} = 0.95$	$\theta^{PD} = 0.95$

5.2 Comparative Statics

The endogenous variables in equilibrium after considering the banker's Bayesian learning behaviour are given in Table 10, and the comparison between the cases with and without the Bayesian effect are given in Table 11 and 12. When the banker does not learn from observing the realization of states, the total mortgage loan at state U is 14.901, while at state D it is 12.002. The negative endowment shock decreases the total mortgage loan by 19.5%, and decreases the Loan-to-Ratio, the mortgage interest rate spread and total deposit by 12.4%, 26.6% and 7.7% respectively. If the banker is a Bayesian updater, who learns about the true state of nature by observing the realization of states, then we can see the endowment shock will have a bigger effect on the equilibrium variables. Given that the true state of nature θ can only take two values, $\theta_1 = 0.99$ and $\theta_2 = 0.91$, which are not too different from the ones in the previous examples, and that the banker's prior belief $\rho = 0.5$, we have $\theta_0^B = 0.95$, $\theta_U^B = 0.952$ and $\theta_D^B = 0.918$. The banker is slightly more optimistic at the good state, and less optimistic at the bad state. Now the negative endowment shock decrease the total mortgage loan, the Loan-to-Value ratio for the mortgage loan, the interest rate spread and the total deposit by 22.3%, 14.3%, 27.3% and 11.7% respectively. It is obvious to see that the effects of the negative endowment shock are amplified when the banker is a Bayesian updater. This is because, when a bad state occurs, the banker's balance sheet shrinks, providing less credit to the economic activities. Meanwhile, the banker becomes less optimistic, and "rationally over-reacts" to the negative endowment shock.

Table 10: Endogenous variables, with a Bayesian learning banker

	0	U	UU	UD	D	DU	DD
M	7.092	13.035	18.565	18.565	13.035	18.939	18.939
π	40.493	43.181			32.412		
ϕ	0.225	0.346			0.358		
A	75.329	76.296			64.964		
$\pi\phi$	9.092	14.940			11.610		
r^{MORT}	0.860	0.767			1.004		
LTV	0.644	0.570			0.489		
Dep	7.092	11.062			9.763		
r	0.838	0.678			0.940		
$r^{MORT} - r$	0.022	0.089			0.064		
P	62.854	75.702	79.391	58.003	66.286	81.136	56.282
c^{R0}	17.021	31.439			28.630		
q^{R0}	0.225	0.111			0.084		
H^{R0}	0.775	0.664			0.691		
c^{P0}	4.979	7.725			3.846		
q^{P0}	0.225	-0.061			0.093		
H^{P0}	0.225	0.163			0.093		
c^{RU}		24.089	40.832	34.035			
q^{RU}		0.174	0.003	-0.113			
H^{RU}		0.826	0.823	0.939			
c^{PU}		10.748	17.335	8.462			
q^{PU}		0.346	0.003	0.233			
H^{PU}		0.346	0.349	0.233			
c^{RD}					21.658	42.517	34.316
q^{RD}					0.142	0.019	-0.118
H^{RD}					0.858	0.838	0.976
c^{PD}					9.866	19.151	8.462
q^{RD}					0.358	0.019	0.241
H^{RD}					0.358	0.378	0.241

Table 11: Bayesian learning effects on financial contracts

	No Bayesian learning			Bayesian learning		
	U	D	%Δ	U	D	%Δ
$\pi\phi$	14.901	12.002	-19.5%	14.940	11.610	-22.3%
LTV	0.570	0.499	-12.4%	0.570	0.489	-14.3%
$r^{MORT} - r$	0.089	0.065	-26.6%	0.089	0.064	-27.3%
Dep	11.020	10.171	-7.7%	11.062	9.763	-11.7%

The banker's learning behaviour has welfare effects as well. As the changes in the beliefs are so small, the welfare effect on the households at state 0 and U are negligible. However, households entering the economy at state D are worse-off if the banker updates the beliefs. The banker becomes less confident in the economy at state D, and thus invests less in the risky assets. The change in optimism amplifies the effect from the shrinking balance sheet, and exacerbates the financial stress for the new house buyers. Minsky's financial instability hypothesis focuses on the building up of risk together with optimism during a prolonged history of prosperity, while this model shows that, when negative shocks happens, the Bayesian updater over-reacts, deteriorating the financial system further. Geanakoplos (2010) elaborates a situation where over-leveraged optimistic traders default and are removed from the economy in bad times, lowering the overall optimism, asset prices, and leverage level. This model demonstrates a similar mechanism, with a shrinking and less optimistic banker in bad times, acting more conservatively.

The results above suggests that, when financial crisis happens, the regulator should consider the change in optimism in the market, in addition to the financial conditions presented in the financial reports.

Table 12: Bayesian learning effects on welfare

No Bayesian learning			Bayesian learning		
$\bar{U}^{R0} = 5.3727$	$\bar{U}^{RU} = 6.3898$	$\bar{U}^{RD} = 6.3935$	$\bar{U}^{R0} = 5.3726$	$\bar{U}^{RU} = 6.3900$	$\bar{U}^{RD} = 6.3914$
$\bar{U}^{P0} = -0.6372$	$\bar{U}^{PU} = 2.1947$	$\bar{U}^{PD} = 2.3182$	$\bar{U}^{P0} = -0.6368$	$\bar{U}^{PU} = 2.1949$	$\bar{U}^{PD} = 2.3164$
$\bar{U}^{B} = 1.9482$			$\bar{U}^{B} = 1.9477$		

6 Conclusion

This paper introduces general equilibrium models with collateral and default to study the mortgage market and the role of financial intermediation. The paper first introduces a mortgage loan contract, and a collateral equilibrium with endogenous default. The model suggests a new method to determine the leverage and interest of the mortgage contract endogenously and simultaneously. Financial intermediation provides risk-sharing and Pareto improves allocation in the housing market. When a negative shock happens to house buyer's endowment, the decreased house price triggers mortgage default, leading to less credit available from the bank for new mortgage loans. Decreased demand and fire sale by the bank further decreases house price, and a downward spiral of credit and prices materializes. Leverage and house price co-moves, due to the change in the banker's balance sheet size. Bayesian learning of the banker exacerbates the effect of a negative endowment shock, and amplifies the financial stress in the housing market.

The paper presents some simple frameworks on mortgage market, credit cycle, and optimism. The recent sub-prime crisis is obviously more complex than how this model describes. Securitization and regulation related issues play major role in the crystallization of the crisis. Our model provides a useful framework for future research on such topics.

Appendices

A First order conditions

A.1 Two households two periods binomial model without bank

$$\begin{aligned}
P_0 U'(c_0^R) &= U'(E - q_0^R) + \beta^R \theta^R \Big(U'(E - q_0^R - q_U^R) + S'(E - q_0^R - q_U^R) \Big) \\
&\quad + \beta^R (1 - \theta^R) \Big(U'(E - q_D^R) + S'(E - q_D^R) \Big) && \text{(A.1)} \\
P_U U'(c_U^R) &= U'(E - q_0^R - q_U^R) + S'(E - q_0^R - q_U^R) && \text{(A.2)} \\
P_D U'(c_D^R) &= U'(E - q_D^R) + S'(E - q_D^R) && \text{(A.3)} \\
\pi U(c_0^R) &= \beta^R \theta^R A U'(c_U^R) + \beta^R (1 - \theta^R) U'(E - q_D^R) && \text{(A.4)} \\
\frac{\partial \pi}{\partial A} U'(c_0^R) &= \beta^R \theta^R U'(c_U^R) && \text{(A.5)} \\
(P_0 - \pi) U'(c_0^P) + \beta^P \theta^P A U'(c_U^R) &= U'(q_0^P) + \beta^P \theta^P \Big(U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P) \Big) && \text{(A.6)} \\
P_U U'(c_U^P) &= U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P) && \text{(A.7)} \\
P_D U'(c_D^P) &= U'(q_D^P) + S'(q_D^P) && \text{(A.8)} \\
\frac{\partial \pi}{\partial A} U'(c_0^P) &= \beta^P \theta^P U'(c_U^P) && \text{(A.9)} \\
\mu = P_0 U'(c_0^P) - U'(q_0^P) - \beta^P \theta^P (U'(q_0^P + q_U^P) &+ S'(q_0^P + q_U^P)) - \beta^P (1 - \theta^P)(U'(q_D^P) + S'(q_D^P)) && \text{(A.10)}
\end{aligned}$$

Consider a power utility function

$$U(x) = \begin{cases} \dfrac{x^{1-\gamma}}{1-\gamma} & \gamma \neq 1 \\ \ln(x) & \gamma = 1 \end{cases}$$

and $S(x) = \tau U(x)$, then we have,

$$\begin{aligned}
\frac{P_0}{(c_0^R)^\gamma} &= \frac{1}{(E - q_0^R)^\gamma} + \frac{\beta^R \theta^R (1 + \tau)}{(E - q_0^R - q_U^R)^\gamma} + \frac{\beta^R (1 - \theta^R)(1 + \tau)}{(E - q_D^R)^\gamma} && \text{(A.11)} \\
\frac{P_U}{(c_U^R)^\gamma} &= \frac{1 + \tau}{(E - q_0^R - q_U^R)^\gamma} && \text{(A.12)} \\
\frac{P_D}{(c_D^R)^\gamma} &= \frac{1 + \tau}{(E - q_D^R)^\gamma} && \text{(A.13)} \\
\frac{\pi}{(c_0^R)^\gamma} &= \frac{\beta^R \theta^R A}{(c_U^R)^\gamma} + \frac{\beta^R (1 - \theta^R) P_D}{(c_D^R)^\gamma} && \text{(A.14)} \\
\frac{\partial \pi}{\partial A} &= \beta^R \theta^R \left(\frac{c_0^R}{c_U^R} \right)^\gamma && \text{(A.15)} \\
\frac{P_0 - \pi}{(c_0^P)^\gamma} + \frac{\beta^P \theta^P A}{(c_U^R)^\gamma} &= \frac{1}{(q_0^P)^\gamma} + \frac{\beta^P \theta^P (1 + \tau)}{(q_0^P + q_U^P)^\gamma} && \text{(A.16)} \\
\frac{P_U}{(c_U^P)^\gamma} &= \frac{1 + \tau}{(q_0^P + q_U^P)^\gamma} && \text{(A.17)} \\
\frac{P_D}{(c_D^P)^\gamma} &= \frac{1 + \tau}{(q_D^P)^\gamma} && \text{(A.18)} \\
\frac{\partial \pi}{\partial A} &= \beta^P \theta^P \left(\frac{c_0^P}{c_U^P} \right)^\gamma && \text{(A.19)}
\end{aligned}$$

thus,

$$c_U^R = \left(\frac{P_U}{1+\tau}\right)^{\frac{1}{\gamma}} (E - q_0^R - q_U^R) \tag{A.20}$$

$$c_D^R = \left(\frac{P_D}{1+\tau}\right)^{\frac{1}{\gamma}} (E - q_D^R) \tag{A.21}$$

$$c_U^P = \left(\frac{P_U}{1+\tau}\right)^{\frac{1}{\gamma}} (q_0^P + q_U^P) \tag{A.22}$$

$$c_D^P = \left(\frac{P_D}{1+\tau}\right)^{\frac{1}{\gamma}} q_D^P \tag{A.23}$$

$$\frac{(q_0+q_U)^\gamma}{(E-q_0-q_U)^\gamma} = \left(\frac{c_U^P}{c_U^R}\right)^\gamma = \left(\frac{c_0^P}{c_0^R}\right)^\gamma = \frac{\frac{1}{(E-q_0)^\gamma} + (1+\frac{A}{P_U})\frac{\beta^R\theta^R(1+\tau)}{(E-q_0-q_U)^\gamma}}{\frac{1}{q_0^\gamma} + (1+\frac{A}{P_U})\frac{\beta^P\theta^P(1+\tau)}{(q_0+q_U)^\gamma}} \tag{A.24}$$

Combined with the budget constraints and collateral constraint, it's easy to see that,

$$P_U = (1+\tau)\left(\frac{e_U^R + e_U^P}{E}\right)^\gamma \tag{A.25}$$

$$P_D = (1+\tau)\left(\frac{e_D^R + e_D^P}{E}\right)^\gamma \tag{A.26}$$

If $\theta^R = \theta^P$ and $\beta^R = \beta^P$, then,

$$q_U = 0 \tag{A.27}$$

A.2 Two binomial mortgage model with a banker

$$P_0U'(c_0^R) = U'(E - q_0^R) + \beta^R\theta^R\left(U'(E - q_0^R - q_U^R) + S'(E - q_0^R - q_U^R)\right)$$
$$+ \beta^R(1-\theta^R)\left(U'(E - q_D^R) + S'(E - q_D^R)\right) \tag{A.28}$$

$$P_UU'(c_U^R) = U'(E - q_0^R - q_U^R) + S'(E - q_0^R - q_U^R) \tag{A.29}$$

$$P_DU'(c_D^R) = U'(E - q_D^R) + S'(E - q_D^R) \tag{A.30}$$

$$U(c_0^R) = \beta^R(1+r)\left(\theta U'(c_U^R) + (1-\theta)U'(c_D^R)\right) \tag{A.31}$$

$$(P_0 - \pi)U'(c_0^P) + \beta^P\theta^P AU'(c_U^R) = U'(q_0^P) + \beta^P\theta^P\left(U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P)\right) \tag{A.32}$$

$$P_UU'(c_U^P) = U'(q_0^P + q_U^P) + S'(q_0^P + q_U^P) \tag{A.33}$$

$$P_DU'(c_D^P) = U'(q_D^P) + S'(q_D^P) \tag{A.34}$$

$$\frac{\partial\pi}{\partial A}U'(c_0^P) = \beta^P\theta^P U'(c_U^P) \tag{A.35}$$

$$\pi(1+r_0)(1+r)\left(\theta^B(1+r_U)U'(Profit_U) + (1-\theta^B)(1+r_D)U'(Profit_D)\right)$$
$$= \theta^B AU'(Profit_U) + (1-\theta)P_DU'(Profit_D) \tag{A.36}$$

$$\theta^B U'(Profit_U) = \frac{\partial\pi}{\partial A}(1+r_0)(1+r)\left(\theta^B(1+r_U)U'(Profit_U) + (1-\theta^B)(1+r_D)U'(Profit_D)\right) \tag{A.37}$$

A.3 Three period model

$$\lambda_1 = (1+r_0^M)(1+r_0)\left((1+r_U^M)\lambda_4 + (1+r_D^M)\lambda_8\right) \tag{A.38}$$

$$\lambda_4 = (1+r_U)\left((1+r_{UU}^M)(\theta^B)^2 U'(Profit_{UU}) + (1+r_{UD}^M)\theta^B(1-\theta^B)U'(Profit_{UD})\right) \tag{A.39}$$

$$\lambda_8 = (1+r_D)\left((1+r_{DU}^M)\theta^B(1-\theta^B)U'(Profit_{DU}) + (1+r_{DD}^M)(1-\theta^B)^2 U'(Profit_{DD})\right) \tag{A.40}$$

$$\pi_0\lambda_1 = A_0\lambda_4 + P_D\lambda_8 \tag{A.41}$$

$$\pi_U\lambda_4 = (\theta^B)^2 U'(Profit_{UU}) + P_{UD}\theta^B(1-\theta^B)U'(Profit_{UD}) \tag{A.42}$$

$$\pi_D\lambda_8 = \theta^B(1-\theta^B)U'(Profit_{DU}) + P_{DD}(1-\theta^B)^2 U'(Profit_{DD}) \tag{A.43}$$

$$\frac{\partial \pi_0}{\partial A_0}\lambda_1 = \lambda_4 \tag{A.44}$$

$$\frac{\partial \pi_U}{\partial A_U}\lambda_4 = (\theta^B)^2 U'(Profit_{UU}) \tag{A.45}$$

$$\frac{\partial \pi_D}{\partial A_D}\lambda_8 = \theta^B(1-\theta^B)U'(Profit_{DU}) \tag{A.46}$$

where λ_1, λ_4 and λ_8 are KKT multipliers.

B More Calculations

B.1 Two period binomial economy with no financial contracts

Seller

$$\begin{aligned}\bar{U}^R &= U(c_0^R, E-q_0^R) + \beta^R\theta^R\left(U(c_U^R, E-q_0^R-q_U^R) + S(E-q_0^R-q_U^R)\right)\\ &\quad + \beta^R(1-\theta^R)\left(U(c_D^R, E-q_0^R-q_D^R) + S(E-q_0^R-q_D^R)\right)\end{aligned} \tag{B.1}$$

$$c_0^R \leqslant e_0^R + P_0 q_0^R \tag{B.2}$$

$$c_U^R \leqslant e_U^R + P_U q_U^R \tag{B.3}$$

$$c_D^R \leqslant e_D^R + P_D q_D^R \tag{B.4}$$

$$\begin{aligned}\bar{U}^P &= U(c_0^P, q_0^P) + \beta^P\theta^P\left(U(c_U^P, q_0^P+q_U^P) + S(q_0^P+q_U^P)\right)\\ &\quad + \beta^P(1-\theta^P)\left(U(c_D^P, q_0^P+q_D^P) + S(q_0^P+q_D^P)\right)\end{aligned} \tag{B.5}$$

$$c_0^P + P_0 q_0^P \leqslant e_0^P \tag{B.6}$$

$$c_U^P + P_U q_U^P \leqslant e_U^P \tag{B.7}$$

$$c_D^P + P_D q_D^P \leqslant e_D^P \tag{B.8}$$

$$q_0^R = q_0^P \tag{B.9}$$

$$q_U^R = q_U^P \tag{B.10}$$

$$q_D^R = q_D^P \tag{B.11}$$

B.2 Bayesian updating beliefs

$$
\begin{aligned}
\theta_U^B &= \theta_1 Prob(\theta=\theta_1|U) + \theta_2 Prob(\theta=\theta_2|U) \\
&= \theta_1 \frac{Prob(U|\theta=\theta_1)Prob(\theta=\theta_1)}{Prob(U|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(U|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&+ \theta_2 \frac{Prob(U|\theta=\theta_2)Prob(\theta=\theta_2)}{Prob(U|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(U|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&= \theta_1 \frac{\theta_1\rho}{\theta_1\rho+\theta_2(1-\rho)} + \theta_2 \frac{\theta_2(1-\rho)}{\theta_1\rho+\theta_2(1-\rho)} \\
&= \frac{\theta_1^2\rho+\theta_2^2(1-\rho)}{\theta_1\rho+\theta_2(1-\rho)}
\end{aligned}
$$

$$
\begin{aligned}
\theta_D^B &= \theta_1 Prob(\theta=\theta_1|D) + \theta_2 Prob(\theta=\theta_2|D) \\
&= \theta_1 \frac{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)}{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&+ \theta_2 \frac{Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)}{Prob(D|\theta=\theta_1)Prob(\theta=\theta_1)+Prob(D|\theta=\theta_2)Prob(\theta=\theta_2)} \\
&= \theta_1 \frac{(1-\theta_1)\rho}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} + \theta_2 \frac{(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
&= \frac{\theta_1(1-\theta_1)p+\theta_2(1-\theta_2)(1-p)}{(1-\theta_1)p+(1-\theta_2)(1-p)}
\end{aligned}
$$

$$
\begin{aligned}
&\theta_U^B - \theta_0^B \\
=&\frac{\theta_1^2\rho+\theta_2^2(1-\rho)}{\theta_1\rho+\theta_2(1-\rho)} - \rho\theta_1 - (1-\rho)\theta_2 \\
=&\frac{\theta_1^2\rho+\theta_2^2(1-\rho)-\rho^2\theta_1^2-(1-\rho)^2\theta_2^2-2\rho(1-\rho)\theta_1\theta_2}{\theta_1\rho+\theta_2(1-\rho)} \\
=&\frac{\rho(1-\rho)(\theta_1-\theta_2)^2}{\theta_1\rho+\theta_2(1-\rho)} > 0 \\
&\theta_0^B - \theta_D^B \\
=&\rho\theta_1+(1-\rho)\theta_2 - \frac{\theta_1(1-\theta_1)\rho+\theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
=&\frac{\rho^2\theta_1(1-\theta_1)+\rho(1-\rho)\theta_1(1-\theta_2)+\rho(1-\rho)\theta_2(1-\theta_1)+(1-\rho)^2\theta_2(1-\theta_2)-\theta_1(1-\theta_1)\rho-\theta_2(1-\theta_2)(1-\rho)}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} \\
=&\frac{\rho(1-\rho)(\theta_1-\theta_2)^2}{(1-\theta_1)\rho+(1-\theta_2)(1-\rho)} > 0
\end{aligned}
$$

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