Signalling with Endogenous Private Information

Matthew Robertson*
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Abstract

We propose a simple and intuitive way to transform canonical signalling games with exogenous types into games in which the informed agent endogenously generates her private information through an unobservable costly effort decision, before attempting to signal this effort. By adapting the seminal set-up of Mailath (1987) and employing the recent equilibrium refinement for such games, Reordering Invariance, we provide general results on the differentiability of action functions and existence of equilibrium. We then apply these results to classic models of the job market and security design to demonstrate the practical utility of endogenous effort. In particular, our approach in these applications lends theoretical support to stylised facts that cannot be derived from the standard signalling framework.

1 Introduction

Signalling games, a class of dynamic Bayesian game, are employed to model situations in which an informed agent tries to communicate some private information to one, or potentially many, uninformed agents. This private information is exogenously endowed, implying the informed agent has no control over her information, and is typically modelled as the realisation of a random variable drawn from some probability distribution. The uninformed agent(s) have knowledge about this probability distribution but do not observe the realisation, and so there is asymmetric information between the two parties. In an attempt to credibly convey her information the informed agent sends an observable signal to the uninformed agents, which is known to be costly for the informed. Uninformed agents then update their beliefs about the informed agent's private information, given the observed signal, before responding in a way that maximises their payoffs. In many applications of signalling games the key states of interest are separating equilibria, equilibria in which the informed agent's private information is completely revealed via the signal. In terms of scope, formal signalling models have been introduced into an extremely wide range of settings including: biology (Grafen, 1990; Smith, 1991); philosophy (Lewis, 1969); and linguistics

*Email: matthew.robertson.100@strath.ac.uk
†University of Strathclyde, Department of Economics, UK.

(Rooy, 1982)[1]. The most widespread utilisation however has, of course, been economics. Since the inception of economic applications in Spence (1973) signalling games have been employed by economists in a diverse assortment of fields: industrial organization (Milgrom and Roberts, 1982); cheap-talk (Crawford and Sobel, 1982); security design (DeMarzo and Duffie, 1999). Most notably, to models of market interactions that have a flavour of adverse selection *à* la Akerlof (1970). However, in many situations involving adverse selection the assumption of exogenous private information is not the most intuitive. Consider a simple used car market, for example, where the quality of any given car can most naturally be thought of as some combination of the current owner's effort towards maintenance and a random aspect representing past owners' level of upkeep. Alternatively consider a firm that underwrites assets before transforming them into securities to sell on secondary markets. The payoff of any particular asset will be determined to a large extent by the effort of the underwriter during ex-ante due-diligence and ex-post monitoring and servicing. Finally, consider workers pondering higher education, each worker's productivity is at least partially engendered through the effort that is dedicated to studying. Therefore, in this paper we introduce endogenous private information, modelled as an unobservable costly effort decision, into the canonical signalling framework developed by George Mailath.

In a seminal contribution Mailath (1987) provided widely applicable conditions under which the informed agent's action function in a signalling game, responsible for mapping types into observable actions, is differentiable. The crucial requirements are that the equilibrium is separating, so that the informed agent's private information is fully revealed by a unique action, and the usual incentive compatibility condition is satisfied, so the informed agent finds it optimal to truthfully reveal her private information given the response each signal generates. If these conditions, and several structural assumptions, are satisfied, Mailath shows that the informed agent's separating action function can be obtained as the solution of the ordinary differential equation implied by incentive compatibility, without resorting to ad-hoc assumptions. These results, both regarding differentiability of action functions and existence of separating equilibria, have been applied in a wide range of settings including: job markets with matching (Hopkins, 2012); cheap talk with lying costs (Kartik, 2008); and altruism (Glazer and Konrad, 1996; Andreoni and Bernheim, 2009). However, two of the structural assumptions, unbounded action spaces and concavity of the informed agent's payoff in her signal, are not satisfied in the classic models of corporate finance and security design proposed by Leland and Pyle (1977) and DeMarzo and Duffie (1999), which feature compact action spaces and linearity of the informed agent's payoff in the signal, respectively. Consequently, Mailath and von Thadden (2013) generalized the results of Mailath (1987) to cover these useful settings, amongst other extensions.

We endogenize private information in the signalling framework of Mailath and von Thadden to answer two main research questions. The first is the characterisation of the action function responsible for mapping each choice of effort into a unique signal. This function is of central importance in applied models of signalling where it characterises the signalling strategy of the

[1]See Sobel (2009) for an in-depth review.

informed agent in the usual separating equilibrium. To provide an answer in a general setting we directly extend a result of Mailath and von Thadden (2013) to our environment in which the informed agent is endowed with an effort production function, which determines a unique output for each effort input, and state the extra assumptions required on this function for the result to hold. Compared to the classic differential equation implied by incentive compatibility in the standard game, the differential equation in this setting takes account of the marginal effect of the informed agent's effort production function in an intuitive way. The second question, and the more important conceptually, can be phrased loosely as, given the behaviour that arises in a separating equilibrium in which the informed agent truthfully signals her private information allowing uninformed agents to correctly respond given their updated beliefs, how does the informed agent choose her effort. More specifically, can the informed agent optimally choose her effort, given the separating behaviour governed by incentive compatibility and the usual zero profit condition, in a manner that is congruous to that seen in producer theory and principal-agent models[2]. To provide a partial answer to this latter question we employ the recent insights of In and Wright (2017), who provide an equilibrium refinement for this class of endogenous signalling games known as Reordering Invariance, and incorporate an endogenous effort choice in the signalling games of two seminal papers: the security design model of DeMarzo and Duffie (1999) and the job market model of Spence (1973). In doing so we provide a natural way to transform the usual exogenous signalling game into one with an endogenous costly effort decision using the framework developed in Mailath (1987) and Mailath and von Thadden (2013). After applying our prior result on the characterisation of the equilibrium signal to these transformed games, we obtain intuitive results as to how the informed agent optimally chooses her effort in the Reordering Invariance equilibria of these two games, and the conditions under which these equilibria exist. Specifically, we show that under noncontroversial assumptions on the informed agent's effort production function and associated disutility, in Spence's model of education the informed agent optimally chooses her effort by equating marginal benefit with marginal cost, and upon introducing a parameter capturing an agent's socioeconomic background, optimal effort is increasing in this measure. Moreover, in the extension of DeMarzo and Duffie, the firm's optimal effort is decreasing in the rate at which it discounts any retained earnings. If the firm is in greater need of liquidity, it underwrites relatively poorer quality assets and subsequently sells a relatively larger quantity of these assets to investors on the secondary market. These results provide theoretical support for two stylised facts often documented in the respective empirical literatures that cannot be derived in the standard exogenous model.

This paper is related to three entwined strands of the literature on the economics of information. First of all to the small but growing literature on signalling games with endogenous choices. The closest paper to ours is the recent work of In and Wright (2017) who provide an equilibrium refinement for this class of games, which often enables one to pin down a unique

[2]Formally, can the problem be set up such that the informed agent's payoff has a form similar to $\max_{K,L}\ \Pi(K,L) = AK^{\alpha}L^{1-\alpha} - rL - wL$ for $\alpha \in (0,1)$ or $\max_e\ U(w,e) = u(w) - c(e)$ where $u'' \leq 0$ and $c'' > 0$.

equilibrium outcome, by appealing to a notion of Reordering Invariance (RI). In particular, they show that when the informed agent does not gain any new payoff relevant information between the two decision nodes, and the uninformed agents strategies and beliefs do not change, the game can be reordered and solved by first considering the choice of signal before effort. This technique also pins down the equilibrium beliefs of the uninformed agents in a way that aligns closely with ex-ante intuition. In and Wright (2016, 2017) study several examples, but without a consistent framework, and often assuming differentiability of the informed agent's action function and existence of the RI-equilibrium. In contrast, we propose a consistent framework, based on that of Mailath (1987) and Mailath and von Thadden (2013), with which to set up applied endogenous signalling games and state the precise conditions required for the action function to be differentiable. Moreover, we develop intuitive results as to how the informed agent optimally chooses her effort in the RI-equilibrium of several seminal signalling games and elucidate the conditions required for these equilibria to exist. This paper also relates to the literature grown out of the two classic signalling games we consider. The paper of DeMarzo and Duffie (1999) has been extended to analyse and explain the following corporate finance phenomena: pooling and tranching of securities (DeMarzo, 2005); screening (Vanasco, 2013); and, the impact of market power taking a mechanism design approach (Biais and Mariotti, 2005). We extend this literature by interpolating a costly effort decision into the model and employing this extension to provide a rationale for the determinants of security payoffs. The original model of Spence (1973) has been greatly extended and merged with other theoretical frameworks by several authors. Hopkins (2012) considers a matching model of the labour market where workers signal productivity via education, whilst Spiewanowski (2010) considers a version of the Spencian universe in which firms observe a noisy signal of a worker's education, and Feltovich et al. (2002) introduce 'medium' productivity types. Finally, Eichberger and Kelsey (1999) consider a version of the model where firms are uncertain over the equilibrium actions of workers, in the sense of Choquet expected utility, and they show that the only equilibrium involves pooling. We endogenize productivity in a continuous version of the Spence model, and use this adaption to offer an explanation of how productivity is engendered when workers use education as a signal, and compare with the situation under symmetric information.

2 Model

The framework we extend is the canonical approach of Mailath (1987), specifically employing the conditions proposed in Mailath and von Thadden (2013). Rather than the informed agent's private information, or type, being exogenously endowed we model the informed agent's private information as being endogenously engendered via an unobservable costly effort decision. This adds an additional stage to the standard signalling model, transforming it into a three stage game. In the first stage the informed agent chooses her unobservable effort that generates a unique outcome. The process by which effort maps into outcomes is common knowledge, but the outcome is known only by the informed agent. In the second stage there is the usual signalling subgame in which the informed agent chooses a second observable action in an attempt to signal her choice of effort in the first stage to the uninformed agent. The uninformed agent cares about the choice of effort of the informed agent, as the particular outcome arising from the choice of effort will impact both his action and payoff, and hence he wants to know the choice of effort to respond optimally. In the third stage the uninformed agent observes the signal and forms beliefs about the chosen effort conditional upon this signal. These beliefs, and any relevant equilibrium conditions such as a zero profit rule, will determine the uninformed agent's best response. This additional stage means that the game may have many sequential equilibria, and the usual method for solving standard exogenous signalling games may not be sufficient for pining down a unique equilibrium. Consequently, In and Wright (2017) propose an equilibrium refinement, Reordering Invariance, which can lead to a unique equilibrium outcome in this class of games. They show that, when the informed agent's information does not change between stages, she will choose her effort with the associated signal in mind. When this requirement is satisfied In and Wright show that, in an RI-equilibrium, the informed agent will rationally choose the same effort and signal pair if the game is reordered so that the second stage is played first; consequently, the informed agent will choose her signalling strategy before her effort. This reordering enables one to use the uninformed agent's strategies and beliefs to shrink the set of equilibria; specifically, In and Wright suggest a form of forward induction for the equilibrium beliefs of the uninformed. This belief formation centres around the uninformed agent making inferences about the costly effort decision by assuming that it was an optimal choice with the observed signal in mind. It is the concept of RI equilibrium that we employ in this paper.

Formally, the informed agent begins by choosing an action, most naturally interpreted as effort, which is observable only by herself, and therefore represents the informed agent's private information. This effort is denoted $\omega \in [\omega_1, \omega_2] \equiv \Omega \subset \mathbb{R}_+$, where Ω is compact. Each effort $\omega \in \Omega$ generates a unique outcome via the $\mathbb{C}^2$ mapping $\varphi : \Omega \to \mathbb{R}_+$, which can be intuitively thought of as a production function. We assume that φ is strictly increasing over the domain of effort except at the left end-point so that $\varphi'(\omega) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. Equally important, we assume that the mapping φ is common knowledge, but the outcome $\varphi(\omega)$ is known only by the informed agent. Furthermore, each $\omega \in \Omega$ generates a corresponding disutility, captured by $\psi : \Omega \to \mathbb{R}_+$, which is assumed to be $\mathbb{C}^2$ and strictly increasing, where $\psi(\omega_1) = 0$ and $\psi'(\omega) > 0$ for all $\omega \in \Omega \setminus \{\omega_1\}$.

After the informed agent selects an unobservable effort, and computes the associated outcome via φ, she takes a second observable action as a means to signal her prior choice of effort to the uninformed agent(s). Specifically, given any $\omega \in \Omega$, the informed agent chooses a signal $x \in \mathcal{X} \subset \mathbb{R}_+$, where $\mathcal{X}$ is compact, by which the uninformed agent forms beliefs about the unobservable ω and its associated payoff $\varphi(\omega)$, via his knowledge of φ. The uninformed agent cares about the chosen effort as, given any signal, he responds with an action $r \in \mathcal{R} \subset \mathbb{R}_+$ through which he aims to maximise his payoff. If the uninformed agent is unable to infer ω he may calculate the incorrect outcome and his subsequent response may not be optimal. Therefore, given any effort ω and subsequent signal x chosen by the informed agent, and response r by the uninformed, we write the informed agent's payoff as $v(x, r, \omega)$. We write the uninformed agent's best response after observing signal x and forming subsequent beliefs $\hat{\omega}$ about the chosen effort, which is used to calculate $\varphi(\hat{\omega})$, as $\rho(x, \varphi(\hat{\omega}))$. This best response determines the optimal r for the uninformed agent, for any signal and belief, and in applications is usually derived from a zero profit condition. Incorporating this best response implies the informed agent's payoff is now defined by the function $V : \Omega^2 \times \mathcal{X} \to \mathbb{R}_+$, which is $\mathbb{C}^2$ in each of its arguements, where

$$V(\omega, \varphi(\hat{\omega}), x) \equiv v(x, \rho(x, \varphi(\hat{\omega})), \omega). \tag{1}$$

We assume that $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0$ for all $x \in \mathcal{X}$ and that $V_2(\omega, \varphi(\hat{\omega}), x) \neq 0$ for all $\hat{\omega} \in \Omega$, where subscripts denote partial derivatives[3]. Note that the assumption on V_3 implies that with symmetric information the problem $\max_{x \in \mathcal{X}} V(\omega, \varphi(\omega), x)$ has a unique solution for each $\omega \in \Omega$ and that this solution, $X^{FB}(\omega)$, lies on the boundary of Ω. This arises as $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0$ implies the informed agent's payoff is strictly decreasing (increasing) in the signal, and as such, when the uninformed can observe the choice of effort, the informed agent's payoff is maximised by not investing (fully investing) in the signal.

After reordering the game however, the informed agent first chooses the observable action. This stage is much like the construction of a separating equilibrium in the standard game, if one exists. As such, the informed agent must find it optimal to use a different signal for each choice of effort, given how the uninformed agent will respond to any credible signal. Formally, the informed agent's signal in this stage of the game, denoted $X : \Omega \to \mathcal{X}$, must be one-to-one so that each $\omega \in \Omega$ maps into a unique signal $X(\omega)$. It must also be the case that, for any ω, the signal must be incentive compatible so that

$$X(\omega) \in \underset{x \in X(\Omega)}{\arg\max} \; V(\omega, \varphi(X^{-1}(x)), x) \tag{IC}$$

holds. As noted, one crucial requirement for a separating equilibria to exist is that the signal, or separating action X, is one-to-one so that $\omega \neq \omega'$ implies $X(\omega) \neq X(\omega')$. Consequently, if such a function exists, any additional properties of X one may be able to elucidate will be of salient importance in applied models of endogenous signalling. One property of particular relevance is differentiability as, if this holds a priori, one can obtain X as the solution to the ordinary differential equation implied by incentive compatibility. Mailath (1987) first provided a set of widely applicable assumptions that, if employed, generate a differentiable observable action function. Whilst these conditions have been successfully applied in many fields they do not apply in this setting, which is akin to the classic security design model of DeMarzo and Duffie (1999). Moreover, the results of Mailath and von Thadden (2013), whilst greatly expanding the possible assumptions that generate a differentiable X to cover this setting, amongst others, do not apply

[3] This last assumption can also be relaxed at one boundary in applications.

here due to the introduction of the informed agent's production technology φ. We extend Mailath and von Thadden's Theorem 3 to include models in which the private information, or type, is endogenously generated through an unobservable effort choice.

Theorem 1. *Let $X : \Omega \to \mathcal{X}$ be one-to-one, where $\mathcal{X}$ is compact, and satisfy (IC). Let $\varphi : \Omega \to \mathbb{R}_+$ be differentiable and one-to-one with $\varphi'(\omega) \neq 0 \;\forall\; \omega \in \Omega$, except at either ω_1 or ω_2. Then, for any $\omega \in \Omega$, if $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0 \;\forall\; x \in \mathcal{X}$, X is differentiable at ω. At all points of differentiability X satisfies*

$$X'(\omega) = -\frac{V_2(\omega, \varphi(\omega), X(\omega)) \cdot \varphi'(\omega)}{V_3(\omega, \varphi(\omega), X(\omega))}\bigg|_{\hat{\omega}=\omega, x=X(\omega).} \tag{DE}$$

Proof. All proofs are in the Appendix. □

Adapting Mailath and von Thadden's theorem to a setting with endogenous private information changes the classic differential equation of separating equilibria to take into account the marginal effect of the informed agent's effort technology. As a result several conceptual and technical assumptions are required on φ. Conceptually, φ must be monotonic, or one-to-one, so that each $\omega \in \Omega$ maps into a unique $\varphi(\omega)$. This ensures that, if the informed agent is able to fully reveal her private information, there is a unique best response for the uninformed; if the production technology is a correspondence, $\varphi : \Omega \rightrightarrows \mathbb{R}_{++}$, this ceases to be the case as each effort no longer generates a unique outcome that the uninformed can respond with. Technically, we require that φ is differentiable for all $\omega \in \Omega$ so that (DE) has a solution, and that $\varphi'(\omega) \neq 0$ for all ω so that $X(\omega) \neq 0$ for all $\omega \in \Omega$. As noted however, this can often be relaxed at one boundary of Ω in applications. Note also that Theorem 1 does not require φ to be increasing or decreasing at this stage, nor do we make assumptions as to its curvature. After computing the signalling strategy for any effort in the reordered game, the informed agent turns to her choice of unobservable costly effort. We write the informed agent's payoff in this stage as a function defined by $\mathcal{V} : \Omega \times \Theta \vee \{1\} \to \mathbb{R}_+$, where

$$\mathcal{V}(\omega, \theta) \equiv V(\omega, \varphi(\omega), X(\omega)),$$

which takes account of the signalling strategy and the uninformed agent's response. The parameter θ will play a key role in determining the optimal choice of effort in applications. Given this set-up, the condition that determines the informed agent's optimal effort is

$$\omega(\theta) \in \underset{\omega \in \Omega}{\arg\max}\; \mathcal{V}(\omega, \theta). \tag{E0}$$

Any solution to (EO) will provide the informed agent a means by which to optimally choose her effort ω, given how she will subsequently signal this effort through $X(\omega)$, and receive response $\varphi(\omega)$ from the uninformed, whose updated beliefs will be confirmed.

Definition 1. The triple of functions $(\varphi(\omega), X(\omega), \omega(\theta))$ and associated beliefs constitute an *Reordering Invariance Equilibrium* if they satisfy the equilibrium condition on the uninformed, (IC), and (EO), respectively.

The following theorem provides one set of sufficient conditions, or recipe, that will lead to the existence of an RI-equilibrium in endogenous signalling games based on the framework and assumptions of Mailath and von Thadden (2013). Note these are not the only set of conditions that will give rise to such an equilibrium. We discuss the conditions present in Theorem 2 in the proof.

Theorem 2. *Given an endogenous signalling game suppose the informed agent's payoff, representable as (1), is linear in the signal and the response of the uninformed, additively separable in effort and the response, concave in effort, and increasing in the response. Finally, suppose that the effort technology is concave. Then a RI-equilibrium exists, $\mathcal{V}_{11}(\omega, \theta) < 0$, if (1) is decreasing in the signal and X is convex/linear (or if (1) is increasing in the signal and X is concave/linear), (1) is additively separable in the signal and the response of the uninformed, and*

$$-\left\{V_{11}(\omega, \varphi(\omega), X(\omega)) + V_2(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}X(\omega)\right\}$$
$$> 2 \cdot V_{13}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}X(\omega). \quad (2)$$

3 Job Market Signalling

In the classic model of job market signalling proposed by Spence (1973) uninformed firms are unable to distinguish between high and low productivity workers. Each worker's productivity is exogenously endowed and private information, whilst the distribution of worker productivities across the population is common knowledge. In the unique separating equilibrium of the game with two levels of productivity, which survives Cho and Kreps's (1987) Intuitive Criterion, the subset of workers endowed by nature with high productivity acquire the minimum education sufficient to incentive low productivity workers not to invest at all. This enables firms to correctly deduce the realised productivity of workers and pay them the correct wage, maximising their payoffs. In Spence (1974) these ideas are extended to setting with a continuum of types, the separating equilibria of which can be loosely interpreted as the many degrees on offer at a higher education institution, and it is in this setting that we endogenize the process by which each worker's productivity is determined. We endogenize productivity as in reality, the productivity of a worker, whilst still potentially being impacted by some random aspect, is largely a function of a workers effort. Specifically, each worker's productivity will be engendered through an unobservable costly effort decision, where each choice of effort generates a unique level of productivity. We seek to answer how, knowing that education may be used as a signal of productivity, workers will choose this unobservable effort that will determine their productivity, which may affect their education decision and the wages paid.

We will show that when each worker takes the behaviour that arises in the separating equilibrium of the signalling game as given, by reordering the game and first computing the education that will fully reveal the choice of effort and maximise the worker's payoff for any effort, and when certain natural assumptions on each worker's production function[4] and effort disutility are satisfied, workers will be able to optimally choose their effort, given how additional effort will impact on their productivity, which feeds through to the choice of education, and the wage they will receive. Intuitively, each worker's effort optimisation will depend on a parameter most naturally interpreted as the worker's socioeconomic background and in the RI-equilibrium of the game, the optimal choice of effort will be increasing in this measure. The intuition is that, while workers with lower socioeconomic backgrounds are able to put in more effort than other workers with higher socioeconomic backgrounds to acquire the same productivity, which is ruled out of the standard exogenous setting, in the RI-equilibrium of the job market signalling game, workers will not find this optimal. Instead, each worker's optimal effort will be uniquely determined by their socioeconomic background, so that workers with different backgrounds will exert different amounts of effort, and subsequently acquire different education signals[5].

After laying out the details of the job market signalling game, and highlighting the differences that arise in this endogenous setting, we apply Theorem 1 to the transformed workers' payoff function to characterise the separating action function that maps effort into education, given incentive compatibility, before turning to analyse optimal effort in this setting. Each worker chooses an unobservable effort $\omega \in [\omega_1, \omega_2] \equiv \Omega$ that determines her productivity $\varphi(\omega)$ via the mapping $\varphi : \Omega \to \mathbb{R}_+$, φ is assumed to be common knowledge, continuously differentiable and strictly increasing. The associated disutility of ω is $\Psi(\omega, \lambda) = \lambda \cdot \psi(\omega)$, which can be thought of as the physical and mental strain of effort in higher education, is also continuously differentiable and strictly increasing in each of its arguements. The parameter $\lambda \in \Lambda$ represents a workers socioeconomic background[6] where a 'smaller' λ implies a lower disutility. After choosing ω each worker chooses a level of observable education $e \in \mathcal{E}$ to obtain as a means to signal her effort, and hence her productivity, to hiring firms[7]. This education comes at cost $c(e, \varphi(\omega))$, most naturally interpreted as the opportunity cost in terms of time spent studying. We assume this cost is submodular in its arguements so that $c_{12}(e, \varphi(\omega)) < 0$, implying the marginal opportunity cost of education is decreasing in effort, and linear in education $c_{11}(e, \varphi(\omega)) = 0$ so that $c_1(e, \varphi(\omega) \neq 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. If a worker is hired by a firm, the firm pays a wage of $r \in \mathcal{R}$. Rather than solving the game by first considering each worker's choice of effort and then subsequently deriving the signal used in the second stage for this effort, we re-order the game so that the worker first calculates the education that allows her to both truthfully reveal her choice of effort and maximise

[4]Responsible for mapping effort into productivity.
[5]A simpler, but less empirically meaningful, interpretation of this parameter is as a measure of each worker's preference for studying.
[6]This may be taken to include parental education, class and number of siblings (Micklewright, 1989), or family income Kohn et al. (1976).
[7]Note that Spence (1974) does not feature compact action spaces, but we feel this assumption can be justified as one's educational choices are in a sense bounded.

her payoff, given the response to this education by firms, for any effort. Each worker then, taking the behaviour previously derived through in the separating equilibrium as given, chooses her effort, knowing how effort feeds into productivity, education and the wage subsequently received. Formally, this implies that each worker will first consider the problem of incentive compatibility for any ω, and then taking the conditions implied by incentive compatibility as given, each worker optimises with respect to effort to find the RI-equilibrium.

Given education e, wage r and effort ω a worker's payoff is written

$$u(e, r, \omega) = r - c(e, \varphi(\omega)) - \lambda\psi(\omega), \tag{3}$$

and the firm's $\pi(r, e, \omega) = \varphi(\omega) - r$. Employing the standard assumption of competition between firms implies a natural zero profit condition, so that in any RI-equilibrium of the signalling game, the firm's payoff is maximised when it best responds with $\rho(e, \varphi(\hat{\omega}))$ given signal e and belief that the chosen effort is $\hat{\omega}$, where

$$\rho(e, \varphi(\hat{\omega})) \equiv r^* - \varphi(\hat{\omega}) = 0.$$

Substituting the action implied by this best response into (3) yields the worker a payoff analogous to (1)

$$U(\omega, \varphi(\hat{\omega}), e) = \varphi(\hat{\omega}) - c(e, \varphi(\omega)) - \lambda\psi(\omega).$$

Given that the game has been re-ordered, we first consider the worker's problem of obtaining the mapping that both truthfully reveals the choice of unobservable effort through the observable signal and maximises the worker's payoff with respect to her education decision. The worker therefore solves

$$E(\omega) \in \underset{e \in E(\Omega)}{\arg\max}\ U(\omega, \varphi(E^{-1}(e)), e). \tag{IC}$$

Assuming for the moment that (IC) is satisfied, so that the worker finds it optimal to use signal $E(\omega)$ when she has chosen effort ω, knowing that the firm will use its knowledge and beliefs to translate the signal into an implied choice of effort that is used to calculate its optimal response, one can characterise the function $E : \Omega \to \mathcal{E}$ by using Theorem 1,

$$E(\omega) = \int_{\Omega} \frac{\varphi'(\hat{\omega})}{c_1(e, \varphi(\omega))}\, \mathrm{d}\omega = \int_{\Omega} \frac{\varphi'(\omega)}{c_1(E(\omega), \varphi(\omega))}|_{e=E(\omega), \hat{\omega}=\omega}\, \mathrm{d}\omega.$$

To obtain a closed form representation of $E(\omega)$, and to make the worker's decision problem of choosing her effort more tractable, we take inspiration from Spence (1973) and assume that $c(e, \varphi(\omega) = e \cdot [\varphi(\omega)]^{-1}$, which satisfies the assumptions of linearity in e and submodularity in (e, ω). Under this assumption $E'(\omega) = \varphi(\omega)\varphi'(\omega)$ and hence $E(\omega) = 1/2 \cdot \varphi(\omega)^2 + k_1$ after integrating by parts, where k_1 is the constant of integration. As in any separating equilibrium, the lowest type, here the left boundary choice of effort, must obtain its first best outcome, so that when $\omega = \omega_1$ we must have $E(\omega_1) = 0$, which implies $k_1 = -1/2 \cdot \varphi(\omega_1)^2$. Together these statements imply that each worker's payoff in the first stage, characterised by (IC) and the zero profit condition imposed on firms, is defined by a function $\mathcal{U} : \Omega \times \Lambda \to \mathbb{R}_+$ where

$$\mathcal{U}(\omega, \lambda) \equiv U(\omega, \varphi(\omega), E(\omega)) = \varphi(\omega) - \underbrace{\frac{1}{2}\left\{\frac{\varphi(\omega)^2 - \varphi(\omega_1)^2}{\varphi(\omega)}\right\}}_{E(\omega)/\varphi(\omega)=c(E(\omega),\varphi(\omega))} - \lambda\psi(\omega). \tag{4}$$

The payoff (4) takes account of the mechanism of separation by which each choice of effort feeds through into a unique level of education, engendering a unique response from the firm. Taking (4) as given, each worker's decision problem is to choose her effort in order to maximise her payoff in the second stage of the reordered game, where workers signal the choice of effort ω with education $E(\omega)$ and receive wage $\varphi(\omega)$. That is, each worker solves

$$\omega(\lambda) \in \arg\max_{\omega \in \Omega} \, \mathcal{U}(\omega, \lambda), \tag{EO}$$

where any solution to this problem is said to be 'effort optimal' (EO). Note that if the solution set of (EO) is empty then signalling may break down, despite (IC) holding[8]. However, before turning to analyse whether the problem (EO) admits a solution, we consider the complete information setting where workers and firms have symmetric information. In such a setting workers acquire no education, as firms are able to observe each worker's choice of effort and compute the relevant productivity, for each choice of effort. This first-best education is denoted $E^{FB}(\omega) = 0 \ \forall \ \omega \in \Omega$. Each worker's effort optimization problem is then $\max_{\omega \in \Omega} \ U(\omega, \varphi(\omega), 0) = \max_{\omega \in \Omega} \ \varphi(\omega) - \lambda\psi(\omega)$, which has unique interior solution $\omega^{FB}(\lambda)$ defined by $\varphi'(\omega) = \lambda\psi'(\omega)$ if $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$ for all $\lambda > 0$. Consequently with complete information, and under standard assumptions on each worker's effort production function and disutility of effort, each worker optimises with respect to effort by following behaviour that equates marginal benefit with marginal cost. Moreover, it is easy to see that the optimal effort under complete information is decreasing in the parameter lambda[9], so that the worker exerts relatively more effort when her socioeconomic background is greater. Turning now to consider the true problem (EO), where the worker takes as given both the education signal E derived in the previous step via (IC) and how the firm will respond φ, we show that under the same relatively mild conditions employed in the complete information setting, and an additional boundary condition on offered employment contracts, workers will continue to be able to optimize by following an analogous effort setting rule.

Proposition 1. *Suppose the opportunity cost of education is given by $c(e, \varphi(\omega)) = e \cdot [\varphi(\omega)]^{-1}$, effort disutility is strictly convex and the effort production function is concave. Suppose further that workers who choose zero effort fall out of the market. Then, in the RI-equilibrium of the*

[8] Informally, without a consistent means of choosing effort workers may end up putting in to much effort relative to their ability and 'burn out'.

[9] Specifically, by the implicit function theorem, we have that $\frac{d}{d\lambda}\omega^{FB}(\lambda) = \frac{\psi'(\omega)}{\varphi''(\omega) - \lambda\psi''(\omega)}$.

job market signalling game characterised by (3) and (IC), for any $\lambda > 0$, there exists a unique effort optimal $\omega^* : \Lambda \to \Omega$ that satisfies (EO) and is a continuous function strictly increasing in socioeconomic background,

$$\frac{d}{d\lambda}\omega^*(\lambda) < 0.$$

Therefore when the effort production function displays weakly decreasing returns to scale, the effort disutility increases at an increasing rate and the productivity that arises from zero effort is also zero, implying no hiring at this effort, $\omega^*(\lambda)$ defined by the implicit solution to $1/2 \cdot \varphi'(\omega) = \lambda\psi'(\omega)$ is sufficient for effort optimality in the RI-equilibrium defined as the tuple $(\varphi(\omega), E(\omega), \omega^*(\lambda))$. The contrast with the symmetric information benchmark is immediate: for any $\lambda > 0$ the marginal benefit of any $\omega \in \Omega \setminus \{\omega_1\}$ in the RI-equilibrium is one half of that under symmetric information.

Remark. *Optimal effort is higher under complete and symmetric information.*

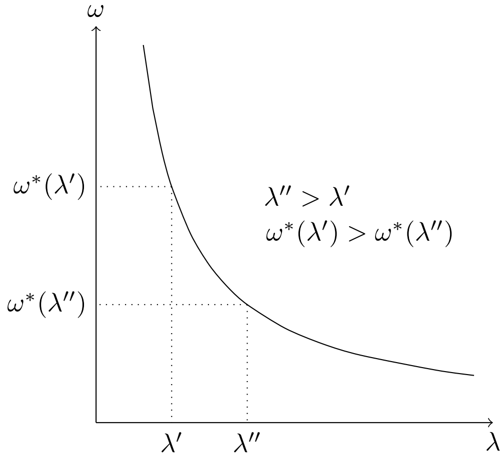
$$\omega^{FB}(\lambda) > \omega^*(\lambda).$$

We have therefore established the following sequence that begins with one parameter and determines two optimal actions progressively with $\lambda \mapsto \omega^*(\lambda) \mapsto E(\omega^*(\lambda))$ as we also will in Section 4. The additional result, gained from endogenising private information, shows that, when workers signal their productivity via education, optimal effort is increasing in socioeconomic background. This is a common finding in the empirical literature on the demand for higher education and human capital theory, which cannot be derived in an exogenous signalling setting. Specifically, James (2002) states that socioeconomic background is largely responsible for a students evaluation of the attainability of higher education in Australia whilst Cameron and Heckman (1998) suggest that, when agents rationally examine the return and costs of higher education, family environment, including level of permanent income, explains a significant amount of the income-schooling relationship. To build some graphical intuition for Proposition 1 we will now illustrate some of the key properties with a simple closed form example.

Example 1. *Suppose that $\Omega = [0, 2]$, $\varphi(\omega) = \omega$ and $\psi(\omega) = \omega^2/2$. Then by Theorem 1 we have $E(\omega) = \frac{1}{2}\omega^2 \ \forall \ \omega \in \Omega$. The worker's payoff is then given by*

$$\mathcal{U}(\omega, \lambda) = \frac{1}{2}\varphi(\omega) - \lambda\psi(\omega) = \frac{1}{2}(\omega - \lambda\omega^2).$$

The problem $\max_{\omega\in\Omega} \frac{1}{2}(\omega - \lambda\omega^2)$ has a unique interior solution $w^*(\lambda) = \frac{1}{2\lambda} \ \forall \ \lambda > 0$.



$\omega^*(\lambda) = \frac{1}{2\lambda}$

It is easy to see that, because optimal effort $\omega^*(\lambda)$ *is defined as the half of the inverse of a worker's socioeconomic background, as* λ *increases,* $\omega^*(\lambda)$ *is decreasing, as per* *Proposition 1. For sufficiency, the second order condition is satisfied as* $\frac{\partial^2}{\partial \omega^2} \frac{1}{2}(\omega - \lambda\omega^2) = -\lambda < 0$*. We illustrate this in* *Figure 1* *with* $\lambda \in \Lambda = \{\lambda_1, \lambda_2\}$ *where* $\lambda_1 = 0.9 > \lambda_2 = 0.6$*.*

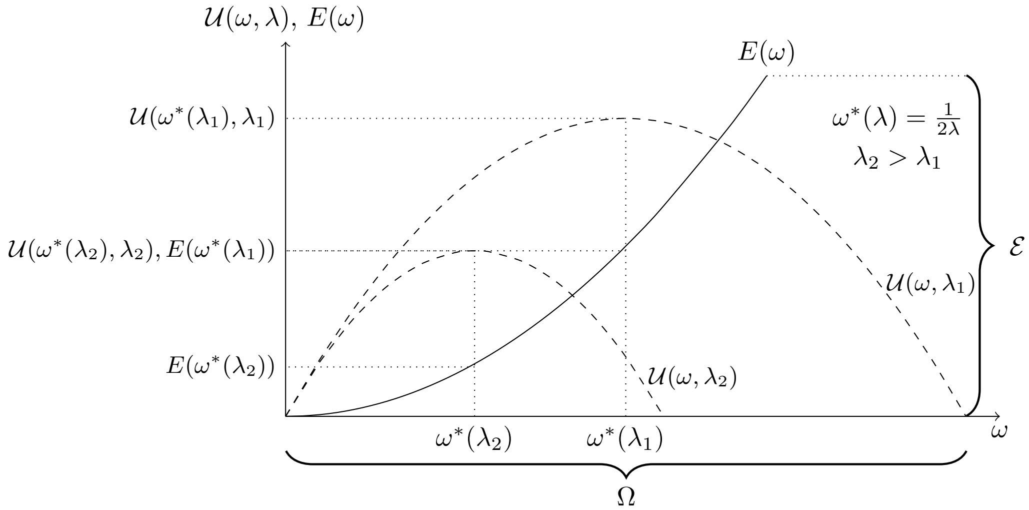


Figure 1: RI-equilibrium in the Spence model

Figure 1 graphically demonstrates that under the conditions postulated in Example 1 the equilibrium payoff function $\mathcal{U}$ is decreasing in λ. To formulate this without recourse to a specific form of effort technology we make use of the envelope theorem for unconstrained optimization. Define the value function $\mathfrak{U}(\lambda) \equiv \max_{\omega \in \Omega} \mathcal{U}(\omega, \lambda) = \mathcal{U}(\omega^*(\lambda), \lambda)$. When the conditions of Proposition 1 hold, we know that $\mathcal{U}$ is concave in effort for all $\omega \in \Omega \setminus \{\omega_1\}$ and that the value of ω that maximises $\mathcal{U}$, given any $\lambda \in \Lambda$ is given by the solution to (EO) and denoted $\omega^*(\lambda)$. Thus, we can compute that $\frac{\partial \mathfrak{U}(\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda}\mathcal{U}(\omega, \lambda)|_{\omega=\omega^*(\lambda)} < 0$, implying the firm's value function $\mathfrak{U}$ is decreasing in λ for all $\lambda \in \Lambda \subset \mathbb{R}_{++}$ and each $\omega \in \Omega \setminus \{\omega_1\}$. In simple terms, when education is primarily used as a signal of productivity, itself engendered through costly effort, it is beneficial to have a socioeconomic background characterised by a relatively large family income and parental encouragement (Mare, 1980).

Finally, consider the setting in which the worker is unable to estimate her exact socioeconomic background before choosing her effort. We represent the distribution of preferences by the cumulative distribution function $F(\lambda|\tau)$, which is parameterized by τ, and hence a worker's expected payoff, given (IC) and (3), is $\mathbb{E}[\mathcal{U}(\omega, \tau)] = \varphi(\omega) - \frac{E(\omega)}{\varphi(\omega)} - \int \Psi(\omega, \lambda) dF(\lambda|\tau)$ where $dF(\lambda|\tau)$ is the density of F. We show in this case that both the optimal choice of effort and the workers expected payoff are decreasing in τ, which could represent the mean of the distribution, when $F(\lambda|\tau)$ is decreasing in this parameter.

Proposition 2. *Suppose that $\tau' > \tau$ implies $F(\lambda|\tau') < F(\lambda|\tau)$. Then $\omega^u(\tau) = \max_{\omega \in \Omega} \mathbb{E}[\mathcal{U}(\omega, \tau)]$ is decreasing in τ. Moreover, $\mathbb{E}[\mathcal{U}(\omega, \tau)] > \mathbb{E}[\mathcal{U}(\omega, \tau')]$.*

More generally, Athey (2002) shows that whenever the density $f(\lambda|\tau) = dF(\lambda|\tau)$ satisfies the monotone likelihood ratio order property, which implies first-order stochastic dominance, and the utility function is supermodular the optimal choice of effort will be weakly increasing in the R-I equilibrium of the Spence model with endogenous effort under uncertainty.[10]

4 Security Design

In DeMarzo and Duffie's model of security design the informed agent, a monopolist issuer of asset-backed securities, has private information related to the payoff of the assets that back the security. The monopolist firm uses this private information to compute the conditional expected value of the security and, once it has performed this computation, it uses the quantity it puts up for sale to signal this expected value to the uninformed agents, market investors. DeMarzo and Duffie assume that the firm's private information is exogenously endowed through the realisation of a random variable, however there are many cases in which the firm that issues the securities is also responsible for underwriting the assets that back each security. In such a setting the firm's private information, and hence the payoff of the security, are engendered endogenously via due-diligence and adequate monitoring and servicing. As such, in this section we extend DeMarzo and Duffie's signalling model to a setting in which the firm chooses an unobservable effort $\omega \in [\omega_1, \omega_2] \equiv \Omega$ that generates a unique monetary payoff $\varphi(\omega)$ at cost $\psi(\omega)$, where it will prove crucial to assume that $\varphi(\omega_1) > 0$. This effort then informs the firm's choice of signal $x \in \mathcal{X} \equiv [0, 1]$, where x is the quantity of the security offered for sale by the firm. The firm discounts the fraction of retained earnings, $(1 - x)$, at rate $\delta \in (0, 1)$, which is strictly less than

[10]Specifically, the monotone likelihood ratio property implies that the density is log-supermodular, as by taking logs of the definition one obtains $\ln(f(\theta^H, s^H)) - \ln(f(\theta^L, s^H)) \geq \ln(f(\theta^H, s^L)) - \ln(f(\theta^L, s^L))$.

the discount rate of market investors. By adapting the model to include an endogenous effort choice we can offer theoretical support for a stylised fact reported by the empirical literature, which states that firms more in need of liquidity securitize a relatively larger amount of their assets and that these assets are of a poorer quality. Our intuitive explanation relates to the means by which firms determine the payoff of assets that they securitize and sell, given that they will signal effort through the quantity offered for sale, and centres on the firms discount factor or preference for liquidity.

Rather than solve the game in the natural order of the moves, we once again reorder the game so that the firm first computes the quantity that it will sell as a signal, for any effort, before turning to the decision problem of choosing effort. The payoff of the firm is set up before we impose a zero profit condition on market investors to obtain their best response to any signal and belief about the chosen effort. This best response enables us to apply Theorem 1 to the firm's payoff to obtain the quantity that allows the firm to truthfully reveal its effort. Subsequently, taking this derived quantity strategy and investors' optimal response as given, we propose a means by which the firm optimally chooses its effort. In particular, when certain conditions are satisfied, optimal effort is born out of the interplay between the firm's discount factor and behaviour defined by equating marginal effort benefit with marginal cost. Under key conditions on the firm's production function, responsible for mapping effort into monetary outcomes, the firm's optimal effort in the RI-equilibrium of the security design signalling game is monotonically decreasing in the discount rate it applies to retained earnings. So that the more the firm discounts retained earnings, relative to the market, the lower is the effort chosen by firms; therefore, echoing the stylised fact.

Hence, given any effort ω, quantity for sale x and price received r, the firm's payoff is written

$$v(x, r, \omega) = \delta(1 - x)\varphi(\omega) + r - \psi(\omega), \tag{5}$$

and market investors' $m(r, x, \omega) = x\varphi(\omega) - r$. Imposing the standard zero profit condition on market investors, $m(r, x, \omega) = 0$ for all $\omega \in \Omega$, we solve for their best response, given signal x and belief that the chosen effort is $\hat{\omega}$,

$$\rho(x, \varphi(\hat{\omega})) \equiv x\varphi(\hat{\omega}) - r^* = 0. \tag{6}$$

We now employ (6) to transform (5) into a form analogous to (1) so that we can apply Theorem 1,

$$\begin{aligned} V(\omega, \varphi(\hat{\omega}), x) &= \delta(1 - x)\varphi(\omega) + x\varphi(\hat{\omega}) - \psi(\omega), \\ &= \delta\varphi(\omega) + (\varphi(\hat{\omega}) - \delta\varphi(\omega))x - \psi(\omega). \end{aligned} \tag{7}$$

In the re-ordered game we first look for a separating action $X : \Omega \to \mathcal{X}$, which is a unique fraction offered for sale for each choice of effort, that satisfies incentive compatibility,

$$X(\omega) \in \underset{x \in X(\Omega)}{\arg\max}\ V(\omega, \varphi(X^{-1}(x)), x). \tag{IC}$$

If (IC) is satisfied then the fraction offered for sale will truthfully reveal the firms choice of effort, allowing market investors to calculate the correct price to pay using their knowledge of φ, and the firm will find this optimal. To derive X in this setting with endogenous private information we apply Theorem 1 to (7), which provides both a means by which to obtain X but also to conclude that X is differentiable. The application of Theorem 1 yields the following differential equation

$$X'(\omega) = -\frac{x \cdot \varphi'(\hat{\omega})}{\varphi(\hat{\omega}) - \delta\varphi(\omega)} = \frac{1}{\delta - 1} \frac{X(\omega) \cdot \varphi'(\omega)}{\varphi(\omega)}\bigg|_{\hat{\omega}=\omega, x=X(\omega)}.$$

We obtain $X(\omega)$ by noting that as a result of Theorem 1 X is differentiable and by employing the boundary condition $X(\omega_1) = X^{FB}(\omega_1) = 1$. This boundary condition arises naturally as when the firm chooses the lowest possible effort $\omega = \omega_1$, where ω_1 is the left endpoint of Ω, it obtains its first best outcome of selling the entire security. The key difference between $X(\omega)$ and the separating action obtained in the standard game is the presence of the firm's effort technology, the assumptions on which greatly impacts the magnitude of the quantity offered for sale for any non boundary effort. Example 2 illustrates this for noncontroversial effort technologies and a suitably defined domain.

Example 2. *Suppose that $\Omega = [1, 2]$ and consider the following two cases: $\varphi_1(\omega) = \omega^{1/2}$ and $\varphi_2(\omega) = \omega$. The firm's payoff in case 1 is*

$$V(\omega, \hat{\omega}^{\frac{1}{2}}, x) = \delta\omega^{1/2} + (\hat{\omega}^{1/2} - \delta\omega^{1/2})x - \psi(\omega). \tag{E1}$$

Applying Theorem 1 to (E1) yields the differential equation

$$X_1'(\omega) = \frac{1}{\delta - 1} \frac{X(\omega) \cdot \frac{1}{2\omega^{1/2}}}{\omega^{1/2}}\bigg|_{\hat{\omega}=\omega, x=X(\omega)} = \frac{1}{2(\delta - 1)} \frac{X(\omega)}{\omega}\bigg|_{\hat{\omega}=\omega, x=X(\omega)},$$

with solution $X_1(\omega) = [\frac{1}{\omega}]^{\frac{1}{2(1-\delta)}} = \omega^{\frac{1}{2(\delta-1)}}$. Repeating the process for the second case obtains $X_2(\omega) = \omega^{\frac{1}{\delta-1}}$. Note that $X_2(\omega)$ is the analogue of the solution found in DeMarzo and Duffie (1999). Figure 2 highlights the shift in magnitude of the fraction offered for sale under a effort production function with decreasing returns to scale.

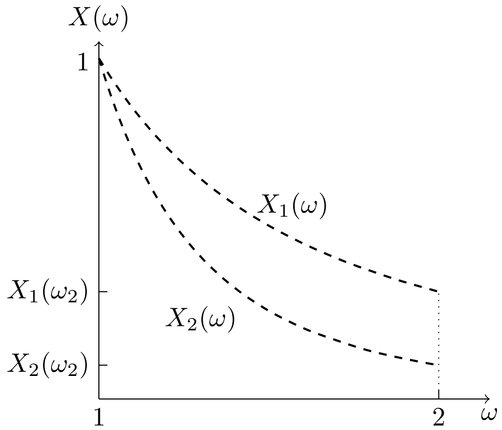


Figure 2: Decreasing vs Constant Returns to Scale

Given (6) and $X(\omega)$ we can write the firm's payoff when it uses the signal defined by X for each effort ω, and subsequently receives price $\varphi(\omega)$ from market investors, as

$$\mathcal{V}(\omega,\delta) \equiv V(\omega,\varphi(\omega),X(\omega)) = \delta\varphi(\omega) + (1-\delta)\varphi(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - \psi(\omega), \tag{8}$$

$$\text{where} \quad X(\omega) = \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}.$$

for any $\omega \in \Omega$. In the re-ordered game the firm now has to choose its effort to maximise (8), which takes account of how effort will impact the retained earnings, the fraction that will be offered for sale, which itself feeds through to the wage received, and the associated disutility. The firm's effort optimisation problem is written

$$\omega(\delta) \in \underset{\omega\in\Omega}{\arg\max}\ \mathcal{V}(\omega,\delta). \tag{EO}$$

As in Section 3 we first consider the problem under complete information, which implies symmetric information between the firm and market investors, before analysing the true problem (EO). In this first-best outcome the firm sells all of the security, $X(\omega)^{FB} = 1$, for every effort, as investors observe the choice of effort and can therefore calculate the relevant monetary outcome. The firm's problem is then $\max_{\omega\in\Omega}\ V(\omega,\varphi(\omega),1) = \max_{\omega\in\Omega}\ \varphi(\omega) - \psi(\omega)$, which provides the unique optimal effort ω^* defined by the solution of $\varphi'(\omega) = \psi'(\omega)$ if $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$. Note that this solution is independent of δ, whilst this parameter, the firm's discount rate, will play a central role under asymmetric information. Specifically, under asymmetric information, and when certain conditions are satisfied, a strictly increasing mapping $\delta \mapsto \omega^*(\delta)$ will exist. This monotonicity enables the RI-equilibrium to exist with endogenous effort. To make this statement we can take two approaches, the first is to make suitable assumptions on the firm's effort production function that ensure that the second-order condition of the problem (EO) is satisfied.

Assumption 1. *The function $\varphi : \Omega \to \mathbb{R}_+$ satisfies*

$$\delta\left[\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - 1\right]\varphi''(\omega) > \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) > 0,$$

for each $\omega \in \Omega \setminus \{\omega_1\}$ and all $\delta \in (0,1)$.

Assumption 1 identifies a class of functions $\varphi \in \mathbb{C}^2$ for which, after taking account of the magnitude of the fraction offered for sale and the discount factor, the second derivative is greater than the derivative of the log of the function. Note that Assumption 1 rules out effort production functions that display constant or increasing returns to scale, intuitively leaving only those functions with decreasing returns to scale. This implies that without strong assumptions on ψ, the firm's payoff is convex in effort under a linear or convex production technology.

Proposition 3. *Suppose the effort production function satisfies Assumption 1 and effort disutility is increasing and strictly convex. Then, in the RI-equilibrium of the security design signalling game characterised by (6) and (IC), for $\delta \in (0,1)$, there exists a unique effort optimal $\omega^* : (0,1) \to \Omega$ satisfying (EO), which is continuous, implicitly defined by*

$$\delta\varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] = \psi'(\omega), \tag{9}$$

and strictly decreasing in a firm's preference for liquidity

$$\frac{d}{d\delta}\omega^*(\delta) > 0.$$

Unlike under complete information, optimal effort implicitly defined by (9) takes account of the firm's discount factor, and as $\delta \in (0,1)$ and $X(\omega) \in [0,1]$, any solution implicitly defined by (9) is strictly less than that defined under complete information, echoing the remark of Section 3. The mechanics of the comparative statics result can be explicated as follows. By (9) the firm's discount rate affects only the firm's marginal effort benefit and hence, holding effort fixed, as the firm's discount rate decreases, the firm's marginal benefit increases whilst marginal cost remains constant; therefore, (9) no longer holds. Consequently, the firm will increase its effort until (9) once again holds with equality, which will be possible if effort disutility rises at a rate greater than that of the effort production function. This increase in effort then increases the monetary value of the asset pool, which causes the firm to sell less of the security to investors; however, this smaller fraction will be sold at a relatively greater price. Knowing this, investors therefore also learn implicitly of the firm's discount rate when they observe a specific retention. The intuition is clear, when the firm's payoff from retaining the assets is very low, perhaps due to a need for liquidity, it has a much larger incentive to exert less effort and securitize a greater fraction of these poorer quality assets to sell to investors, as the relatively low price will be preferred to retention. Conversely, when the firm does not discount retained earnings to such a degree, it is more willing to spend time and care on the due-diligence process knowing that this will feed through to a relatively smaller amount sold to investors, but at a relatively greater price. To provide conceptual support for the intuition of Proposition 3 we briefly draw on empirical evidence to informally motivate how a market comprised of N firms may behave in a setting with endogenous private information. Suppose that N is finite, that each $i \in \{1, \dots, N\}$ has discount factor δ_i, where $\delta_i \neq \delta_j$ for $i \neq j$ and $\Delta = \{\delta_1, \dots, \delta_N\}$, and that Assumption 1 holds. Moreover, suppose that the effort production function and disutility, φ_i and ψ_i, are symmetric so that $\varphi_i = \varphi_j = \varphi \; \forall \; i$ and similarly for ψ. Proposition 3 then implies that the partial ordering $(\Delta, <)$ uniquely determines both the partial ordering of effort by each firm and the subsequent partial ordering of quantities offered for sale. Specifically, $\delta_i > \delta_j$ implies that $\omega^*(\delta_i) > \omega^*(\delta_j)$, which in turn implies that $X(\omega^*(\delta_i)) < X(\omega^*(\delta_j))$. Therefore, firms with relatively high discount rates will underwrite assets of relatively lower value than those firms that have relatively low discount rates. The firms with assets with relatively lower payoffs will then securitize a relatively larger proportion of these assets to sell to market investors. If we interpret δ_i as firm i's preference for liquidity, which can arise due to more profitable investment opportunities, or a need to satisfy capital adequacy requirements, this informal discussion of the implications of Proposition 3 is supported empirically. Cardone-Riportella et al. (2010) show that generally firms with relatively low liquidity will have relatively lower performance and subsequently securitize more and Bannier and Hansel (2008) support this finding, as they provide evidence that banks with lower liquidity, greater credit risk exposure and worse performance measures are more likely to securitize and sell a larger proportion of their assets. Martin-Oliver and Saurina (2007) and Agostino and Mazzuca (2009) find that the key motivating factor behind bank securitization in Spanish and Italian banks, respectively, over the period 1999-2006 was a need for liquidity. Finally, Affinito and Tagliaferri (2010) find that banks that are less profitable, less liquid and with more troubled loans are more likely to securitize assets and at a larger quantity than otherwise.

Unfortunately, Assumption 1 is restrictive and difficult to check in practice without first fixing a value of δ. To circumvent this assumption we can turn to the tools of monotone comparative statics and instead make use of the property of increasing differences. The usual assumption when applying Topkis's Theorem is that (8) satisfy increasing differences in the choice variable and parameter of interest, here the firm's effort and its discount factor. We begin with a preliminary result that demonstrates that this assumption is unnecessary as intuitively this property arises endogenously in this setting.

Lemma 1. *$\mathcal{V}(\omega, \delta)$ is strictly supermodular and differentiable on $\Omega \setminus \{\omega_1\} \times (0, 1)$ so that $\mathcal{V}_{12}(\omega, \delta) > 0$.*

Hence, for any fixed effort, the marginal effect of effort on $\mathcal{V}$ is increasing in the firm's discount factor. This key condition, and $\mathcal{X}$ being a compact real interval, allows us to apply Edlin and Shannon (1998)'s Strict Monotonicity Theorem and Corollary 1 to (8). As a result we can conclude that, if the problem (EO) has an interior solution, by Lemma 1 $\omega^*(\delta) = \max_{\omega \in \text{int}(\Omega)} \mathcal{V}(\omega, \delta)$

is strictly increasing in δ. Despite the supermodularity arising endogenously, the conditions present in this setting are stronger than those sufficient for applying Topkis's Theorem. Namely, rather than arguing that the argmax correspondence is increasing in the strong set order, in this differentiable setting, inherited from assumptions on the firm's payoff, effort production function, and the signal, given Theorem 1, we can applying Edlin and Shannon's stronger result for a unique solution strictly increasing in δ. The assumption that $\mathcal{V}$ attains an interior maximum is less restrictive than concavity but still enforces some structure that may not be inherently present.

Example 3. *Suppose that $\Omega = [0, 2]$, $\psi(\omega) = 0$ and take $\delta = 1/2$. Then the firm's payoff with costless effort is written*

$$\mathcal{V}(\omega, \delta) = \delta\varphi(\omega)[1 + X(\omega)]. \tag{E2.}$$

Supposing further that $\varphi(\omega) = 1 + \omega^{1/2}$ and applying Theorem 1 to $V(\omega, 1 + \hat{\omega}^{1/2}, x)$ yields $X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}}$, which we substitute into (E2.) to give

$$\mathcal{V}(\omega, 1/2) = \left[1 + \left(\frac{1}{\varphi(\omega)}\right)^{\frac{1}{1-\frac{1}{2}}}\right]\frac{\varphi(\omega)}{2} = \left[1 + \left(\frac{1}{\varphi(\omega)}\right)^{2}\right]\frac{\varphi(\omega)}{2} = \frac{1 + \varphi(\omega)^2}{2\varphi(\omega)} = \frac{1 + (1 + w^{1/2})^2}{2(1 + \omega^{1/2})}.$$

Partially differentiating $\mathcal{V}(\omega, 1/2)$ twice with respect to ω obtains

$$\mathcal{V}_{11}(\omega, 1/2) = -\frac{1}{8}\frac{w^2 + 3w^{3/2}}{w^2(1 + w^{1/2})^3} < 0 \ \forall \ \omega \in \Omega \setminus \{\omega_1\}.$$

$\mathcal{V}(\omega, 1/2)$ with $\psi(\omega) = 0$ is illustrated in Figure 3.

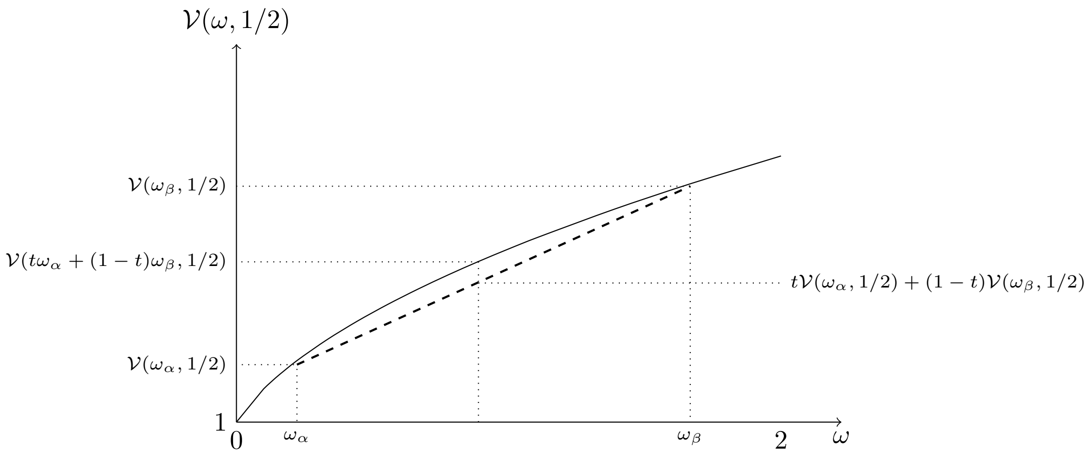


Figure 3: $\psi(\omega) = 0$

Thus, when we include convex costs of effort ($-\psi''(\omega) < 0$) we will have a strengthening of the concavity of $\mathcal{V}$ as the sum of a concave function and a strictly concave function is strictly *concave. In Figure 4 we illustrate $\mathcal{V}(\omega,\delta)$ with $\psi(\omega)=w^2/2$, which implies $\psi''(\omega)>0$, for $\delta\in\Delta\equiv\{\delta_1,\delta_2\}$, where $\delta_2=0.8>\delta_1=0.5$, and the associated quantities $X(\omega^*(\delta_i))$, $i\in\{1,2\}$.*

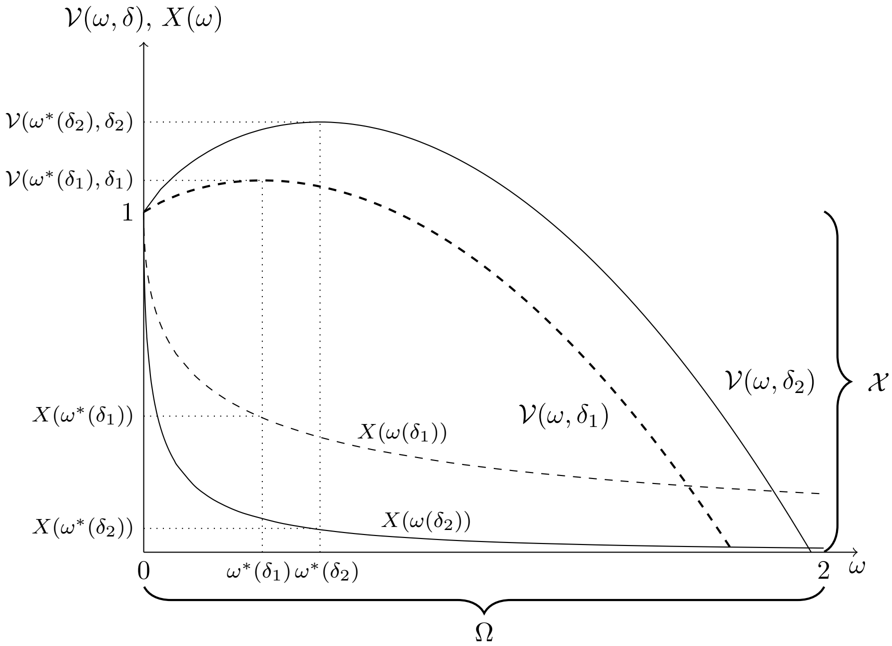


Figure 4: $\psi''(\omega)>0$

Whilst the firm's value function, $\mathfrak{V}(\delta)\equiv\max_{\omega\in\Omega}\ \mathcal{V}(\omega,\delta)=\mathcal{V}(\omega^*(\delta),\delta)$, is clearly increasing in Figure 4 this is not true in general, unlike in Section 3. Specifically, the firm's value function is increasing, $\frac{\partial\mathfrak{V}(\delta)}{\partial\delta}=\frac{\partial}{\partial\delta}\mathcal{V}(\omega,\delta)|_{\omega=\omega^*(\delta)}>0$, when

$$(1-\delta)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]>-\ln\left(\frac{\varphi(\omega_1)}{\varphi(\omega)}\right)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}, \tag{10}$$

and decreasing otherwise. Condition (10), whilst admittedly abstract, holds for effort production functions such as $\varphi(\omega)=\ln(\omega)+1$ and $\varphi(\omega)=1-\frac{1}{2}e^{-x}$ for $\Omega\subset\mathbb{R}_+$, or $\varphi(\omega)=\ln(\omega)$ for $\omega_1>1$, which satisfy the intuitive requirement of being increasing and concave, as well as $\varphi(\omega)=1+\omega^{1/2}$ as in Example 3. Note that the right hand side of (10) can be written $-X(\omega)[\ln(\varphi(\omega_1))-\ln(\varphi(\omega))]$. This more stringent condition arises because the signal depends also on the parameter δ, unlike Section 3 where the signal is independent of the relevant parameter. Finally, we again consider the case when the firm does not know the value of its discount factor, and instead knows only its distribution $G(\delta|\tau)$ parameterized by τ. The firm's expected profit prior to choosing effort is then

$$\mathbb{E}[\mathcal{V}(\omega,\tau)]=\varphi(\omega)\int\delta dG(\delta|\tau)+\varphi(\omega)\int(1-\delta)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}dG(\delta|\tau)-\psi(\omega).$$

By following similar methods to the proof of Proposition 2 one can show that if $\tau' > \tau$ implies $G(\delta|\tau') \leq G(\delta|\tau)$, then both the firm's optimal choice of effort, $\omega^u(\delta) = \max_{\omega\in\Omega} \mathbb{E}[\mathcal{V}(\omega, \tau)]$, and the firm's expected profit, are increasing in τ. The key difference in the proof compared to Proposition 2 consists of using the fact that the left-hand side of the first-order condition is positive, implying that the terms involving τ form an increasing function of ω, to show that the first-order stochastic dominance inequality holds in this case.

5 Conclusion

In this paper we propose a simple and intuitive way to transform canonical signalling games with exogenous types into three stage games in which the informed agent first endogenously generates the analogue of her type though an unobservable costly effort decision, before attempting to signal her effort in the second stage. Our method involves adapting the classic framework of Mailath (1987), subsequently developed in Mailath and von Thadden (2013), and employing the recent equilibrium refinement propounded by In and Wright (2016, 2017). Their equilibrium refinement, Reordering Invariance, allows us to reorder the game and to first solve the conventional signalling subgame before turning to the choice of optimal unobservable effort.

There are several motivations for extending signalling games in this way. The first is that in many cases, such as the two applications considered in Section 3 and Section 4, it is more natural to model the informed agent's private information as arising endogenously via an unobservable effort choice rather than being exogenously endowed; therefore, more closely approximating reality. The second salient motivation is that by replacing exogenous types with endogenous hidden actions we increase the explanatory power of signalling games to encompass a natural account as to how informed agents would, if given the choice, choose their type in games where this private information is signalled via an observable action. Moreover, we demonstrate that, under certain conditions, this form of choice is consistent with the type of behaviour normally observed in separating equilibria.

Our main results are closely related to this motivation. The first set, Theorem 1 and Theorem 2, provide a characterisation of the informed agent's signalling strategy that takes account of her newly endowed effort technology, directly extending a result of Mailath and von Thadden (2013), and a recipe for setting up an endogenous signalling game that will have a unique RI-equilibrium, respectively. This recipe is one set of, relatively restrictive, sufficient conditions for the existence of RI-equilibria and therefore provides a simple first step towards characterising when such an equilibrium exists. Specifically, we focus on the linear setting of Mailath and von Thadden (2013) and so one further step could be to extend Theorem 2 to the concave setting of Mailath (1987).

The second set of results, Proposition 1, Proposition 2 and Proposition 3, highlight the additional insight gained by endogenising private information, which we illustrate in two seminal applications: Spence's (1973) job market model; and DeMarzo and Duffie's (1999) model of security design. In particular, we show that the informed agent's optimal unobservable effort in the security design setting, in which the firm signals its effort[11] using the quantity of the security offered for sale, is decreasing in the firms need for liquidity. Conversely, the worker's optimal effort is increasing in her socioeconomic background when investment in education is used as a signal of effort, which itself determines the worker's productivity. These intuitive results provide theoretical support for two stylised facts often documented in the respective empirical literatures that cannot be derived in standard signalling games and demonstrates the practical utility of endogenous private information in such games. In principle, results of this form should be obtainable in any differentiable signalling game by introducing our form of endogenous effort and any relevant assumptions, which can then be critiqued in terms of applicability and intuition as is done with Assumption 1.

[11] Implying that, since the effort production function is common knowledge, the firm implicitly signals the payoff of the security through the signal of its effort. This holds analogously in the model of the job market.

6 Appendix

Proof of Theorem 1. Deriving the preliminary results of Mailath and von Thadden (2013) with our modified framework yields the following expression analogous to Mailath and von Thadden's (A5):

$$0 \geq g(\omega_0, \varphi(\omega), X(\omega)) \geq -(\omega - \omega_0)\Big\{\frac{1}{2}g_{11}([\omega;\lambda])_1(\omega-\omega_0) \\ + g_{12}([\omega;\mu])_{23}(\varphi(\omega)-\varphi(\omega_0)) + g_{12}([\omega;\mu])_{23}(X(\omega)-X(\omega_0))\Big\}, \quad \text{(A1)}$$

where

$$[\omega;\lambda]_1 \equiv (\lambda\omega_0 + (1-\lambda)\omega, \varphi(\omega), X(\omega)),$$

$$[\omega;\mu]_{23} \equiv (\omega_0, \mu\varphi(\omega_0) + (1-\mu)\omega, \mu X(\omega_0) + (1-\mu)X(\omega)),$$

and

$$g(\omega, \varphi(\hat{\omega}), x) \equiv V(\omega, \varphi(\hat{\omega}), x) - V(\omega, \varphi(\omega_0), X(\omega_0))$$

for fixed $\omega_0 \in \Omega$, arbitrary $\omega, \hat{\omega} \in \Omega$, $\lambda \in [0,1], \mu \in [0,1]$ and $x \in \mathcal{X}$. The simple yet key change in (A1) relative to Mailath and von Thadden's (A5) is the presence of the informed agent's effort technology, which alters the g_{12} term.

Performing a Taylor series expansion on $g(\omega_0, \varphi(\omega), X(\omega))$ around $(\omega_0, \varphi(\omega_0), X(\omega_0))$ and simplifying,

$$g(\omega_0, \varphi(\omega), X(\omega)) = g_2(\omega_0, \varphi(\omega_0), X(\omega_0))(\varphi(\omega)-\varphi(\omega_0)) + g_3(\omega_0, \varphi(\omega_0), X(\omega_0))(X(\omega)-X(\omega_0)) \\ + g_{22}([\omega;\gamma])_{23}(\varphi(\omega)-\varphi(\omega_0))^2 + g_{33}([\omega;\gamma])_{23}(X(\omega)-X(\omega_0))^2 \\ + g_{23}([\omega;\gamma])_{23}(\varphi(\omega)-\varphi(\omega_0))(X(\omega)-X(\omega_0)). \quad \text{(A2)}$$

Substituting (A2) into (A1),

$$0 \geq \\ g_2(\omega_0, \varphi(\omega_0), X(\omega_0))(\varphi(\omega)-\varphi(\omega_0)) + g_3(\omega_0, \varphi(\omega_0), X(\omega_0))(X(\omega)-X(\omega_0)) \\ + g_{22}([\omega;\gamma])_{23}(\varphi(\omega)-\varphi(\omega_0))^2 + g_{33}([\omega;\gamma])_{23}(X(\omega)-X(\omega_0))^2 + g_{23}([\omega;\gamma])_{23}(\varphi(\omega)-\varphi(\omega_0))(X(\omega)-X(\omega_0)) \\ \geq \\ -(\omega-\omega_0)\Big\{\frac{1}{2}g_{11}([\omega;\lambda])_1(\omega-\omega_0) + g_{12}([\omega;\mu])_{23}(\varphi(\omega)-\varphi(\omega_0)) + g_{12}([\omega;\mu])_{23}(X(\omega)-X(\omega_0))\Big\},$$

and dividing through by $(\omega - \omega_0)$,

$$0 \geq \\ g_2(\omega_0, \varphi(\omega_0), X(\omega_0))\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0} + g_3(\omega_0, \varphi(\omega_0), X(\omega_0))\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0} \\ + g_{22}([\omega;\gamma])_{23}\frac{(\varphi(\omega)-\varphi(\omega_0))^2}{\omega-\omega_0} + g_{33}([\omega;\gamma])_{23}\frac{(X(\omega)-X(\omega_0))^2}{\omega-\omega_0} + g_{23}([\omega;\gamma])_{23}\frac{(\varphi(\omega)-\varphi(\omega_0))(X(\omega)-X(\omega_0))}{\omega-\omega_0} \\ \geq \\ -\frac{1}{2}g_{11}([\omega;\lambda])_1(\omega-\omega_0) - g_{12}([\omega;\mu])_{23}(\varphi(\omega)-\varphi(\omega_0)) - g_{12}([\omega;\mu])_{23}(X(\omega)-X(\omega_0)). \quad \text{(A3)}$$

Finally, taking limits as ω approaches ω_0 from above, $\omega \searrow \omega_0$, on (A3)

$$0 \geq g_2(\omega_0, \varphi(\omega_0), X(\omega_0)) \lim_{\omega\searrow\omega_0}\left[\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0}\right] + g_3(\omega_0, \varphi(\omega_0), X(\omega_0)) \lim_{\omega\searrow\omega_0}\left[\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0}\right] \geq 0,$$

implying that X is differentiable at ω_0 and

$$g_2(\omega_0, \varphi(\omega_0), X(\omega_0)) \lim_{\omega\searrow\omega_0}\left[\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0}\right] + g_3(\omega_0, \varphi(\omega_0), X(\omega_0)) \lim_{\omega\searrow\omega_0}\left[\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0}\right] = 0.$$

We have therefore shown that when $V_3(\omega_0, \varphi(\omega_0), X(\omega_0)) \neq 0$ and when X is continuous at ω_0 that X is differentiable at this point and the derivative satisfies

$$X'(\omega_0) = -\frac{V_2(\omega_0, \varphi(\omega_0), X(\omega_0)) \cdot \varphi'(\omega_0)}{V_3(\omega_0, \varphi(\omega_0), X(\omega_0))}.$$

To then show that if $\omega \longrightarrow \omega_0$ then $V(\omega_0, \varphi(\omega_0), X(\omega)) \longrightarrow V(\omega_0, \varphi(\omega_0), X(\omega_0))$ the proof follows Mailath and von Thadden's Lemma D and so will not be reproduced. The key point to note is that the proof continues to go through as we have assumed that φ is continuous and therefore, for each $\epsilon > 0$ and $\omega \in \Omega$, there is a $\delta > 0$ such that

$$\omega_0 \in \Omega \text{ and } |\omega - \omega_0| < \delta \Rightarrow |\varphi(\omega) - \varphi(\omega_0)| < \epsilon,$$

and hence we still have, as required,

$$|\omega - \omega_0| < \delta \implies |V(\omega_0, \varphi(\omega), x) - V(\omega_0, \varphi(\omega_0), x)| < \epsilon.$$

Having shown convergence, the final piece is the continuity of X at ω_0. For this Mailath and von Thadden's proof of Theorem 3 continues to apply where they employ the compactness of $\mathcal{X}$, the assumption on V_3, and the Bolzano-Weierstrass Theorem to find a convergent subsequence within $\mathcal{X}$ that demonstrates continuity of X at ω_0. □

Proof of Theorem 2. Before showing that, given (IC), the problem (EO) has a solution, we first demonstrate that (IC) continues to hold in this setting with endogenous effort. This first portion of the proof of Theorem 2 follows Mailath and von Thadden's (2013) Lemma H. Incentive compatibility requires that the following first order condition holds,

$$\frac{\partial}{\partial x} V(\omega, \varphi(X^{-1}(x)), x) = V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_3(\omega, \varphi(X^{-1}(x)), x) = 0. \quad \text{(A4)}$$

By differentiating (A4) we obtain the second-order sufficient condition

$$\begin{aligned} \frac{\partial^2}{\partial x^2} V(\omega, \varphi(X^{-1}(x)), x) &= V_{22}(\omega, \varphi(X^{-1}(x)), x) \cdot \left(\varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x)\right)^2 \\ &+ V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d^2}{dx^2} X^{-1}(x) + V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi''(X^{-1}(x)) \cdot \left(\frac{d}{dx} X^{-1}(x)\right)^2 \\ &+ 2 \cdot V_{23}(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_{33}(\omega, \varphi(X^{-1}(x)), x). \quad \text{(A5)} \end{aligned}$$

To show that (A5) is negative we first evaluate (A4) at $\omega = X^{-1}(x)$,

$$V_2(X^{-1}(x), \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_3(X^{-1}(x), \varphi(X^{-1}(x)), x) = 0.$$

Differentiating this identity yields

$$\begin{aligned} &\Big(V_{12}(X^{-1}(x), \varphi(X^{-1}(x)), x) + V_{22}(X^{-1}(x), \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x))\Big) \cdot \left(\frac{d}{dx} X^{-1}(x)\right)^2 \cdot \varphi'(X^{-1}(x)) \\ &+ V_2(X^{-1}(x), \varphi(X^{-1}(x)), x) \cdot \left[\varphi''(X^{-1}(x)) \cdot \left(\frac{d}{dx} x^{-1}(x)\right)^2 + \varphi'(X^{-1}(x)) \cdot \frac{d^2}{dx^2} X^{-1}(x)\right] \\ &+ \Big(V_{13}(X^{-1}(x), \varphi(X^{-1}(x)), x) + 2 \cdot V_{23}(X^{-1}(x), \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x))\Big) \cdot \frac{d}{dx} X^{-1}(x) \\ &+ V_{33}(X^{-1}(x), \varphi(X^{-1}(x)), x) = 0. \quad \text{(A6)} \end{aligned}$$

Re-arranging (A6), and dropping $(X^{-1}(x), \varphi(X^{-1}(x)), x)$ from the notation,

$$V_2 \cdot \varphi''(X^{-1}(x)) \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2 + V_2 \cdot \varphi'(X^{-1}(x)) \cdot \frac{d^2}{dx^2}X^{-1}(x) + V_{33}$$
$$=$$
$$-\left(V_{12} + V_{22} \cdot \varphi'(X^{-1}(x))\right) \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2 \cdot \varphi'(X^{-1}(x)) - \left(V_{12} + 2 \cdot V_{23} \cdot \varphi'(X^{-1}(x))\right) \cdot \frac{d}{dx}X^{-1}(x). \quad \text{(A7)}$$

Evaluating (A5) at $\omega = X^{-1}(x)$,

$$\frac{\partial^2}{\partial x^2}V(X^{-1}(x), \varphi(X^{-1}(x)), x) = V_{22} \cdot \left(\varphi'(X^{-1}(x)) \cdot \frac{d}{dx}X^{-1}(x)\right)^2 + 2 \cdot V_{23} \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx}X^{-1}(x)$$
$$+ V_2 \cdot \varphi''(X^{-1}(x)) \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2 + V_2 \cdot \varphi'(X^{-1}(x)) \cdot \frac{d^2}{dx^2}X^{-1}(x) + V_3. \quad \text{(A8)}$$

Substituting (A7) into (A8) and simplifying gives,

$$\frac{\partial^2}{\partial x^2}V(X^{-1}(x), \varphi(X^{-1}(x)), x) = -V_{12} \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2 \cdot \varphi'(X^{-1}(x)) - V_{13} \cdot \frac{d}{dx}X^{-1}(x),$$
$$= -\left(\frac{d}{dx}X^{-1}(x)\right)^2 \left(V_{12} \cdot \varphi'(X^{-1}(x)) + V_{13} \cdot X'(\omega)\right). \quad \text{(A9)}$$

Therefore (A5) is satisfied when $\varphi'(\cdot)$ is increasing, which we have assumed, as Mailath and von Thadden show that $V_{12} + V_{13} \cdot X'(\omega) > 0$. We obtain (A9) by employing the property of the derivative of inverse functions,

$$\frac{d}{dx}X^{-1}(x) = \frac{1}{X'(\omega)} = X'(\omega) \cdot \frac{1}{X'(\omega)^2} = X'(\omega) \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2.$$

Finally, note that if $V_{12} = 0$, then (A9) is the same condition obtained as in the exogenous setting.

To now show that, given (IC), the problem (EO) has a solution we begin with the form suggested by (1), namely $V(\omega, \varphi(\hat{\omega}), x)$, which becomes

$$\mathcal{V}(\omega, \theta) \equiv V(\omega, \varphi(\omega), X(\omega)) = V(\omega, \varphi(\hat{\omega}), x)|_{\hat{\omega}=\omega, x=X(\omega)},$$

after (IC) based optimization. Hence, for a RI-equilibrium to exist it is sufficient to check that

$$\mathcal{V}_{11}(\omega, \theta) = \frac{\partial^2}{\partial \omega^2}V(\omega, \varphi(\omega), X(\omega) \leq 0 \ \forall \ \omega \in \Omega,$$

which yields the following derivative

$$\mathcal{V}_{11}(\omega, \theta) = V_{11}(\omega, \varphi(\omega), X(\omega)) + V_2(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) + V_{22}(\omega, \varphi(\omega), X(\omega)) \cdot \left(\frac{d}{d\omega}\varphi(\omega)\right)^2$$
$$+ V_3(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}X(\omega) + V_{33}(\omega, \varphi(\omega), X(\omega)) \cdot \left(\frac{d}{d\omega}X(\omega)\right)^2$$
$$+ 2 \cdot V_{12}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}\varphi(\omega) + 2V_{13}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}X(\omega)$$
$$+ 2 \cdot V_{23}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}\varphi(\omega) \cdot \frac{d}{d\omega}X(\omega),$$

as by continuity $V_{12} = V_{21}$, $V_{13} = V_{31}$ and $V_{23} = V_{32}$. By the assumption of linearity in both the signal and the response of uninformed, and additive separability in the informed agent's effort and the response of the uninformed we have

$$\mathcal{V}_{11}(\omega,\theta) = \underbrace{V_{11}(\omega,\varphi(\omega),X(\omega)) + V_2(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}X(\omega)}_{<0}$$
$$+\,2\cdot V_{13}(\omega,\varphi(\omega),X(\omega))\cdot\frac{d}{d\omega}X(\omega) + 2\cdot V_{23}(\omega,\varphi(\omega),X(\omega))\cdot\frac{d}{d\omega}\varphi(\omega)\cdot\frac{d}{d\omega}X(\omega). \quad \text{(A10)}$$

So far the conditions we have employed are relatively uncontroversial. Linearity in the signal has been a central assumption in the two applications considered in Section 3 and 4. Concavity of the informed agent's payoff, before optimization, holds in both applications as

$$\frac{\partial^2}{\partial\omega^2}V(\omega,\varphi(\hat{\omega}),x) = \delta\varphi''(\omega)(1-x) - \psi''(\omega) \le 0,$$

in our extension of DeMarzo and Duffie and,

$$\frac{\partial^2}{\partial\omega^2}U(\omega,\varphi(\hat{\omega}),e) = e\frac{\varphi''(\omega)}{\varphi(\omega)^2} - 2e\frac{\varphi'(\omega)^2}{\varphi(\omega)^3} - \lambda\cdot\psi''(\omega) \le 0,$$

in the extension of Spence. Moreover, additivity of the informed agent's payoff (1) is satisfied in effort and the response of the uninformed will be satisfied in general, and indeed is in our applications. For example, any model where the response of the uninformed takes the form of a wage or price etc will lead to (1) being increasing in this response and additively separable in effort and price/wage, which combined with concave/linear effort technology and our other statements leads to the first three terms of (A10) being negative.

The set of conditions required for the theorem are somewhat more restrictive. To make the conclusion of existence one possible 'recipe' is to assume that $V_{23} = 0$, implying additive separability of (1) in signal and the response of the uninformed, which is satisfied in Section 3 but not in Section 4, which gives

$$\mathcal{V}_{11}(\omega,\theta) = V_{11}(\omega,\varphi(\omega),X(\omega)) + V_2(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}X(\omega)$$
$$+\,2\cdot V_{13}(\omega,\varphi(\omega),X(\omega))\cdot\frac{d}{d\omega}X(\omega).$$

This simplifying assumption removes the complication of the product of terms $\varphi'(\omega)\cdot X'(\omega)$ that may not be negative. Unfortunately, the following expression holds

$$V_{13}(\omega,\varphi(\omega),X(\omega)\cdot\frac{d}{d\omega}X(\omega) > 0, \quad \text{(A11)}$$

for all signalling models. The reason is that V_{13} captures increasing (decreasing) differences in the informed agent's objective in effort and signal, which in turn plays a role in determining whether the equilibrium signal is increasing (decreasing) in effort. Hence, if $V_{13} > 0$ then $X'(\omega) > 0$ and (A11) holds, whilst if $V_{13} < 0$ then $X'(\omega) < 0$ and we are once again left with (A11). Hence, using (2) we can conclude that

$$-\left\{V_{11}(\omega,\varphi(\omega),X(\omega)) + V_2(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega,\varphi(\omega),X(\omega))\cdot\frac{d^2}{d\omega^2}X(\omega)\right\}$$
$$> 2\cdot V_{13}(\omega,\varphi(\omega),X(\omega))\cdot\frac{d}{d\omega}X(\omega) \;\Rightarrow\; \mathcal{V}_{11}(\omega,\theta) < 0.$$

□

Proof of Proposition 1. Under the conditions of Proposition 1 with $c(e,\varphi(\omega) = e\cdot\varphi(\omega)^{-1}$, the worker's payoff is given by (4), which is

$$\mathcal{U}(\omega,\lambda) = \varphi(\omega) - \frac{1}{2}\left\{\frac{\varphi(\omega)^2 - \varphi(\omega_1)^2}{\varphi(\omega)}\right\} - \lambda\psi(\omega).$$

If the worker chooses zero effort, and we assume that zero effort yields zero productivity, then $\varphi(\omega_1) = 0$. Thus,

$$\begin{aligned}\mathcal{U}(\omega,\lambda) &= \varphi(\omega) - \frac{1}{2}\frac{\varphi(\omega)^2}{\varphi(\omega)} - \lambda\psi(\omega),\\ &= \varphi(\omega) - \frac{\varphi(\omega)}{2} - \lambda\psi(\omega),\\ &= \frac{\varphi(\omega)}{2} - \lambda\psi(\omega).\end{aligned}$$

It is easy to see that under the conditions postulated in Proposition 1, where the effort production function is concave and disutility strictly convex, so that $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$, the function $\mathcal{U}(\omega,\lambda)$ is now the sum of a concave function and a strictly concave function, and is therefore strictly concave. Explicitly, the second partial derivatives are,

$$\mathcal{U}_{11}(\omega,\lambda) = \frac{\varphi''(\omega)}{2} - \lambda\psi''(\omega) < 0 \quad \text{when} \quad \varphi''(\omega) < 0,$$

and

$$\mathcal{U}_{11}(\omega,\lambda) = -\lambda\psi''(\omega) < 0 \quad \text{when} \quad \varphi''(\omega) = 0,$$

for $\lambda > 0$. To obtain the comparative statics result, we first define an identity, using the first-order necessary condition,

$$\mathcal{U}_1(\omega,\lambda) = 0 \;\Rightarrow\; \frac{\varphi'(\omega)}{2} = \lambda\psi'(\omega),$$

that implicitly defines optimal effort $\omega(\lambda)$, which is continuous by continuity of φ, ψ and $\lambda > 0$,

$$Z(\omega(\lambda),\lambda) \equiv \frac{\varphi'(\omega(\lambda))}{2} - \lambda\psi'(\omega(\lambda)) = 0,$$

to which we can apply the implicit function theorem, as the strict concavity of $\mathcal{U}$ in ω has been demonstrated, to yield

$$\frac{d}{d\lambda}\omega^*(\lambda) = -\frac{-\psi'(\omega(\lambda))}{\frac{\varphi''(\omega(\lambda))}{2} - \lambda\psi''(\omega(\lambda))} = \frac{\psi'(\omega(\lambda))}{\frac{\varphi''(\omega(\lambda))}{2} - \lambda\psi''(\omega(\lambda))} < 0 \quad \text{where} \quad \varphi''(\omega) < 0,$$

and

$$\frac{d}{d\lambda}\omega^*(\lambda) = -\frac{-\psi'(\omega(\lambda))}{-\lambda\psi''(\omega(\lambda))} = \frac{\psi'(\omega(\lambda))}{-\lambda\psi''(\omega(\lambda))} < 0 \quad \text{where} \quad \varphi''(\omega) = 0,$$

as $\psi'(\omega) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$.

□

Proof of Proposition 2. We will show submodularity of $\mathbb{E}[\mathcal{U}(\omega,\tau)]$ in (ω,τ) by first showing showing supermodularity of $\int \Psi(\omega,\lambda)dF(\lambda|\tau)$ in (ω,τ). Note that $\Psi_1(\omega,\lambda) = \lambda\psi'(\omega) > 0$. The required inequality for $\omega' > \omega$ and $\tau' > \tau$ is

$$\begin{aligned}\int \Psi(\omega',\lambda)dF(\lambda|\tau') - \int \Psi(\omega,\lambda)dF(\lambda|\tau') &\geq \int \Psi(\omega',\lambda)dF(\lambda|\tau) - \int \Psi(\omega,\lambda)dF(\lambda|\tau),\\ \int [\Psi(\omega',\lambda) - \Psi(\omega,\lambda)]dF(\lambda|\tau') &\geq \int [\Psi(\omega',\lambda) - \Psi(\omega,\lambda)]dF(\lambda|\tau). \qquad \text{(A12)}\end{aligned}$$

Note that

$$[\Psi(\omega',\lambda) - \Psi(\omega,\lambda)] = \lambda\psi(\omega') - \lambda\psi(\omega) > 0,$$

as ψ is increasing. Therefore as $F(\lambda|\tau') < F(\lambda|\tau)$ by assumption (A12) holds by first-order stochastic dominance. To show submodularity of the objective in (ω,τ) we multiply (A12) by -1, as the rest of the objective in independent of τ, to give

$$-\int [\Psi(\omega',\lambda) - \Psi(\omega,\lambda)]dF(\lambda|\tau') \leq -\int [\Psi(\omega',\lambda) - \Psi(\omega,\lambda)]dF(\lambda|\tau).$$

□

Proof of Proposition 3. To prove Proposition 3, we first show that, under Assumption 1, $\mathcal{V}(\omega,\delta)$ is concave in it's first argument, and, by demonstrating sufficiency, we apply the implicit function theorem. Prior to this however, we will derive $X : \Omega \to \mathcal{X}$. Beginning with the differential equation featured in the text:

$$X'(\omega) = -\frac{x \cdot \varphi'(\omega)}{\varphi(\hat{\omega}) - \delta\varphi(\omega)} = \frac{X(\omega) \cdot \varphi'(\omega)}{(\delta - 1)\varphi(\omega)}\bigg|_{\hat{\omega}=\omega, x=X(\omega)}.$$

After re-arranging we arrive at the following linear first order ordinary differential equation

$$X'(\omega)(1-\delta)\varphi(\omega) + X(\omega)\varphi'(\omega) = 0. \tag{A13}$$

To solve (A13) we first separate the variables $\frac{X'(\omega)}{X(\omega)} = \frac{1}{\delta-1}\frac{\varphi'(\omega)}{\varphi(\omega)}$, and by integrating this expression one obtains

$$\int \frac{X'(\omega)}{X(\omega)}\,\mathrm{d}\omega = \frac{1}{\delta-1}\int \frac{\varphi'(\omega)}{\varphi(\omega)}\,\mathrm{d}\omega.$$

By applying integration by parts we have

$$\ln(X(\omega)) = \ln(\varphi(\omega)^{\frac{1}{\delta-1}}) + c_1,$$

which can be solved for our equilibrium mapping by noting that

$$\ln(\varphi(\omega)^{\frac{1}{\delta-1}}) + c_1 = \ln(e^{\ln(\varphi(\omega)^{\frac{1}{\delta-1}}+c_1)}) = \ln(e^{\ln(\varphi(\omega)^{\frac{1}{\delta-1}})}e^{c_1}) = \ln(\varphi(\omega)^{\frac{1}{\delta-1}}e^{c_1}),$$

and hence

$$X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}}c_2 \quad \text{where} \quad c_2 = e^{c_1}. \tag{A14}$$

To derive the boundary condition that will enable an explicit form for c_2, note that for $\omega = \omega_1$ the firm will choose $X(\omega_1) = 1$, the first best outcome: $X^{FB}(\omega_1)$. Inserting this into (A14) yields $c_2 = \varphi(\omega_1)^{\frac{1}{1-\delta}}$, which is then substituted back into (A14) to give

$$X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}}\varphi(\omega_1)^{\frac{1}{1-\delta}} = \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}.$$

Now simplifying the firm's payoff in the second stage, which recall is given by (8),

$$\begin{aligned}
\mathcal{V}(\omega,\delta) &= \delta\varphi(\omega) + (1-\delta)\varphi(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - \psi(\omega),\\
&= \delta\varphi(\omega) + (1-\delta)\varphi(\omega)^{1+\frac{1}{\delta-1}}\varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi(\omega),\\
&= \delta\varphi(\omega) + (1-\delta)\varphi(\omega)^{\frac{\delta}{\delta-1}}\varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi(\omega).
\end{aligned} \tag{A15}$$

Partially differentiating (A15) with respect to the firm's choice of effort obtains,

$$\begin{aligned}
\mathcal{V}_1(\omega,\delta) &= \delta\varphi'(\omega) + \frac{\delta}{\delta-1}(1-\delta)\varphi(\omega)^{\frac{\delta}{\delta-1}-1}\varphi(\omega_1)^{\frac{1}{1-\delta}}\varphi'(\omega) - \psi'(\omega),\\
&= \delta\varphi'(\omega) + \delta\left(\frac{1-\delta}{\delta-1}\right)\varphi(\omega)^{\frac{1}{\delta-1}}\varphi(\omega_1)^{\frac{1}{1-\delta}}\varphi'(\omega) - \psi'(\omega),\\
&= \delta\varphi'(\omega) - \delta\left(\frac{\delta-1}{\delta-1}\right)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\varphi'(\omega) - \psi'(\omega),\\
&= \delta\varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega).
\end{aligned} \tag{A16}$$

The first-order condition (A16) is a necessary condition for the firm's choice of effort to satisfy (EO). To check sufficiency we partially differentiate again to obtain,

$$\mathcal{V}_{11}(\omega,\delta) = \delta\varphi''(\omega) - \delta\varphi(\omega_1)^{\frac{1}{1-\delta}} \times \left\{\varphi''(\omega)\varphi(\omega)^{\frac{1}{\delta-1}} + \varphi'(\omega)\left(\frac{1}{\delta-1}\right)\varphi(\omega)^{\frac{1}{\delta-1}-1}\varphi'(\omega)\right\} - \psi''(\omega).$$

This second-order condition simplifies as follows

$$\mathcal{V}_{11}(\omega,\delta) = \delta\varphi''(\omega) - \delta\varphi''(\omega)\varphi(\omega_1)^{\frac{1}{1-\delta}}\varphi(\omega)^{\frac{1}{\delta-1}} + \frac{\delta}{1-\delta}(\varphi'(\omega))^2\varphi(\omega)^{\frac{1}{\delta-1}-1}\varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi''(\omega),$$

$$= \delta\varphi''(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] + \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) - \psi''(\omega). \tag{A17}$$

From (A17) the following preliminary result is immediate.

Lemma 2. *Suppose the effort production function is linear or convex, and the effort disutility is linear. Then, $\mathcal{V}(\omega,\delta)$ is convex in effort.*

Proof of Lemma 2. Linear production and disutility implies that (A17) reduces to

$$\varphi'(\omega)\frac{\delta}{1-\delta}X(\omega) \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) > 0 \quad \text{for} \quad \varphi'(\omega) > 0,$$

whilst convex production implies that (A17) is positive. □

Consider now the case with a concave effort production function whilst continuing to assume that effort disutility is linear, $\psi''(\omega) = 0$. In this case we have

$$\mathcal{V}_{11}(\omega,\delta) < 0 \iff \delta\varphi''(\omega)\left[\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - 1\right] > \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega),$$

as per Assumption 1. Finally note that we can write

$$\mathcal{V}_{11}(\omega,\delta) \overset{\text{sgn}}{=} (\delta-1)\left[\left(\frac{\varphi(\omega)}{\varphi(\omega_1)}\right)^{\frac{1}{1-\delta}} - 1\right]\mathcal{R}(\omega) + \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) \quad \text{where} \quad \mathcal{R}(\omega) = -\frac{\varphi''(\omega)}{\varphi'(\omega)}.$$

Given that we have shown that Assumption 1 is sufficient for $\mathcal{V}(\omega,\delta)$ to be concave in effort, we can now apply the implicit function theorem to yield the comparative statics result. Define

$$Z(\omega(\delta),\delta) \equiv \delta\varphi'(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega(\delta)) = 0.$$

Differentiating the identity $Z(\omega(\delta),\delta)$ yields

$$\frac{d}{d\delta}Z(\omega(\delta),\delta) = \frac{\partial}{\partial\omega}\left(\delta\varphi'(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega(\delta))\right)\frac{d}{d\delta}\omega(\delta) + \varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] = 0,$$

hence

$$\frac{d}{d\delta}\omega(\delta) = -\frac{\frac{\partial}{\partial\delta}\left(\delta\varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right]\right)}{\frac{\partial}{\partial\omega}\left(\delta\varphi'(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega(\delta))\right)},$$

$$= -\frac{\varphi'(\omega) - \left[\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} + \delta\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\ln\left(\frac{\varphi(\omega_1)}{\varphi(\omega)}\right)\frac{d}{d\delta}\frac{1}{1-\delta}\right]}{\delta\varphi''(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] + \varphi'(\omega(\delta))\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega(\delta))](\omega(\delta)) - \psi''(\omega(\delta))},$$

$$= -\frac{\varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] - \frac{\delta}{(1-\delta)^2}\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\left[\ln(\varphi(\omega_1)) - \ln(\varphi(\omega))\right]}{\delta\varphi''(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] + \varphi'(\omega(\delta))\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega(\delta))](\omega(\delta)) - \psi''(\omega(\delta))}. \tag{A18}$$

We can express this in the more compact form

$$\frac{d}{d\delta}\omega(\delta) = \frac{\varphi'(\omega)\Big[X(\omega) - 1\Big] + \sigma(\delta)\varphi'(\omega)X(\omega)\Big[\ln(\varphi(\omega_1)) - \ln(\varphi(\omega))\Big]}{\delta\varphi''(\omega)X(\omega) + (1-\delta)\sigma(\delta)\varphi(\omega)X(\omega) \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega)},$$

$$\text{where} \quad \sigma(\delta) = \frac{\delta}{(1-\delta)^2}.$$

Note that (A18) implies

$$\mathcal{V}_{12}(\omega,\delta) = \varphi'(\omega)\Big[1 - X(\omega)\Big] - \sigma(\delta)\varphi'(\omega)X(\omega)\Big[\ln(\varphi(\omega_1)) - \ln(\varphi(\omega))\Big], \tag{A19}$$

which to sign we note that we have two cases: the first is when $0 < \varphi(\omega_1) < \varphi(\omega) < 1$ and the second is when $\varphi(\omega) > 1$. In the first case we have that

$$\ln(\varphi(\omega_1)) - \ln(\varphi(\omega)) < 0 \quad \text{as} \quad \ln(\varphi(\omega_1)) < \ln(\varphi(\omega)) < 0,$$

and in the second assume that $\varphi(\omega_1) = 1$, without loss of generality as $\ln(\cdot)$ is increasing, so that

$$\ln(1) - \ln(\varphi(\omega)) < 0 \quad \text{as} \quad \ln(1) = 0 \quad \text{and} \quad \ln(\varphi(\omega)) > 0 \text{ for } \varphi(\omega) > 1.$$

Hence we can conclude that (A19) is positive as required for supermodularity and Lemma 1, $\mathcal{V}_{12}(\omega,\delta) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. □

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