Debt constraints and monetary policy*

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Abstract

In the present paper we show how simple monetary policies can mitigate real effects of credit frictions. We consider stationary overlapping generations economies in which consumers are not equally efficient in producing capital and cannot commit to repay loans. At real equilibria, both less productive and more productive consumers engage in capital production. At monetary equilibria consumers can hold money instead of engaging in capital production. Money in itself does not mitigate the real effects of credit frictions and equilibrium allocations are generally not Pareto optimal. However printing money and distributing them to young consumers increase their incomes allowing young more productive consumers to produce more capital. Thereby output is increased.

Keywords Financial Frictions · Growth · Monetary Policy

JEL Classification D5 · E4 · E5 · O4

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1 Introduction

Overview of paper Simple monetary policies in form of printing money can mitigate the effects of frictions in credit markets in situations like the financial crisis where agents have limited access to credit. How money transfers are distributed is important for the effectiveness of the policies. In the present paper we study economies in which consumers cannot commit to repay loans and examine the effects on output and welfare of monetary policies in form of printing money and distributing them to agents.

We consider overlapping generations (OG) economies with production. There are four commodities: a consumption good, capital, labor and money. Consumers live for two periods, consume both when young and old, and work when young. Moreover they produce capital by use of the consumption good. Some consumers are more productive than others in production of capital. Firms produce the consumption good by use of capital and labor. Productivities of consumers are common knowledge. Therefore from an efficiency point of view less productive consumers should leave capital production to more productive consumers.

We assume that there is a friction in the credit market: consumers cannot commit to repay loans. Since there are only two dates, usual penalties such as exclusion from credit markets are ineffective. Hence debt is constrained to zero. Consequently there is no transfer of resources between consumers in the same generation so both kinds of consumers can very well be investing in capital production. Printing money and distributing them to more productive consumers enable these consumers to buy consumption goods from less productive consumers. Consequently more productive consumers produce more capital and less productive consumers hold money. Therefore printing money enables consumers to ease the effects of the debt constraints.

Markets are competitive: consumers maximize their utilities and firms their profits, both given prices. At real equilibria, where money has no value, consumers can save solely by investing in capital production. Therefore all consumers engage in capital production. In Theorem 1 we show that real equilibria exist. Since consumers can solely save by engaging in capital production in real equilibria, there can be over-accumulation of capital leading to dynamically inefficient equilibrium allocations. At monetary equilibria, where money has value, consumers can save by holding money as well as by engaging in capital production. The mere existence of money as an alternative means of saving can be welfare improving provided there is over-accumulation of capital because it crowds out investment in capital production. Steady states are equilibria where growth rates of consumption and production are constant across dates. In Theorem 2 we show that both monetary and real steady states exist.

The existence of money in itself does not help to mitigate the real effects of credit constraints. However simple monetary policies in form of printing money and distributing them solely to young more productive consumers or to all young consumers can reduce the effects. For given prices monetary policies increase the income of consumers who receive transfers and therefore increase their savings provided consumption when old is a normal good. Unsurprisingly printing money decreases the return on money. Indeed if too much money is printed, then the return on money is lower than the return on investment in capital production for less productive consumers so money becomes worthless. In Theorem 3 we characterize monetary policies for which monetary steady states exist. Monetary policies have several effects on consumers: additional income for consumers who receive transfers as well as higher wage and lower return on monetary savings for both kinds of consumers independent of whether they receive transfers or not. It is easily seen that simple monetary policies increase output. We do not consider social policies aimed at distributing welfare gains evenly, but in Theorem 4 we show that a wide range of allocations can be implemented by use of monetary policies involving both money transfers and money taxes.

Unless the returns on money and on investment in capital production for more productive consumers are equal, the two kinds of consumers face different returns on savings. Therefore equilibrium allocations are generally not weakly Pareto optimal. Indeed it is possible to improve welfare by changing consumption at any two consecutive dates without changing production at any date. In general monetary policies in form of printing money and distributing it to young more productive consumers cannot make returns equal. However these policies are simple and mitigate the real effects of debt constraints. Monetary policies involving money taxes eliminate the need for printing money, but enforcement of tax payment is needed. Therefore institutions that can ensure payment of taxes have to be available.

Our results in Theorems 1–4 are obtained under reasonably general assumptions. In Section 6 we study an example to further illustrate our findings as well as the welfare implications of monetary policies. In the example we consider a wider range of monetary policies, where both money transfers and money taxes are allowed, including policies where all young consumers receive identical transfers and all old consumers receive identical transfers.

Related literature Balasko and Shell (1980, 1981) provide the first thorough analysis of general pure exchange economies with many consumption goods per date and many consumers per generation. Weakly Pareto optimal and Pareto optimal allocations as well as bona fide monetary policies are characterized.

There is a large literature on the economic effects of endogenous debt limits. Kehoe and Levine (1993) study pure exchange optimal growth economies where consumers cannot commit to repay loans. Defaulting consumers are excluded from future trading in financial markets, but not from future trading in spot markets. Azariadis and Lambertini (2003) consider pure exchange OG economies with a single good and three dates of lives for consumers who cannot commit to repay loans. Exclusion from financial markets may be a punishment for consumers defaulting when middle-aged, because they may need financial markets to transfer income from when middle-aged to when old. Since exclusion is not a punishment for consumers defaulting when old, middle-aged consumers can lend, but not borrow. Consequently there are non-trivial debt limits for young consumers, but not for middle-aged consumers. It is found that endogenous debt limits can result in multiple steady states and indeterminacy.

Azariadis and Kaas (2008) consider endogenous growth economies with a single good and infinitely lived consumers and endogenous debt limits. Consumers have access to production with stochastic transformation rates so consumers with access to inferior production should not produce. Defaulting consumers can be forced to pay a fraction of output. Conditions on enforceability of repayment of loans ensuring first-best equilibrium growth are provided. Intuitively less discounting of future consumption implies less enforceability is needed. In addition it is shown that endogenous debt limits can result in multiple equilibria. Farhi and Tirole (2012) consider OG economies with production and consumers living for three periods. For investment and production, consumers may need to borrow but only a fraction of future output can be offered as collateral. The conditions for a bubbly equilibrium and the interplay of bubbles and other types of liquidity are studied.

The papers closest to the present study are probably Martin and Ventura (2012, 2016) where OG economies with production and bubbles are considered. Some consumers are more productive than others in production of capital. Old bubbles may die and new bubbles arise, both happens exogenously. There are crowding-out as well as crowding-in effects on investment: old bubbles crowd out investment and new bubbles crowd in investment. Policies in form of subsidies and taxes on credit to stabilize bubbles are studied. In our economies, there is money and monetary policies in form of money transfers and money taxes are possible. Money is an example of a bubble, but it is a bubble that is controllable by monetary policies and not subject to random birth or death. Consequently we are able to study monetary policies and their effects. We use a version of Martin and Ventura (2012) as our example in Section 6.

Antinolfi et al. (2016) consider pure exchange optimal growth economies with two commodities, namely a consumption good and money, and two types of consumers: credit agents who can save by use of debt and money and cash agents who can save solely by use of money. A planner can issue IOUs with a rate of return less than one to cash agents and use the profit from IOUs to offer consumption smoothing to credit agents. It is found that there is an optimal rate of return on IOUs and this is equivalent to a non-trivial optimal rate of inflation.

In the New Monetarist literature search models are used to evaluate monetary policies. Gu et al. (2016) find that money and credit are perfect substitutes in those models and that this finding is robust to variations in policy, debt limit and use of collateral. In our economies, the presence of money is not a perfect remedy for endogenous debt constraints, and the presence of perfectly functioning credit markets would not necessarily eliminate dynamic inefficiency of equilibrium allocations. Consequently money and credit are not perfect substitutes.

The mainstream central banking literature based on the New Keynesian dynamic stochastic general equilibrium models with sticky prices explains why inflation targeting is a natural objective (Woodford 2003). In the absence of aggregate shocks, usually zero inflation is recommended in order to minimize the consequences of sticky prices. In our economies prices are flexible and inflation can be desirable provided money taxes are not possible. Moreover, for a given inflation target the real effects of monetary policy depend on the distribution of money across consumer types.

Plan of the paper In Section 2 our model is introduced. In Section 3 equilibria and steady states are defined and studied for economies without money printing. In Section 4 the equilibrium dynamics are studied. In Section 5 the effects of printing money and distributing them to young more productive consumers are studied. In Section 6 an example is considered and the effects of monetary policies consisting of printing money and distributing them to young consumers, both more productive and less productive, are worked out. Section 7 contains some final remarks.

2 The model

We consider stationary overlapping generations economies with endogenous growth. Our economies have two distinctive features: consumers are not equally efficient in producing capital; and, consumers cannot commit to repay loans. In the present section we introduce our setup and examine the decision problems of the agents.

Setup

Time is discrete and extends from $-\infty$ to $+\infty$. At every date there are a consumption good, capital, labor and possibly money. Let $(P_t, R_t, W_t) \in \mathbb{R}^3_{++}$ be the nominal prices of consumption, capital and labor at date t.

At every date a continuum of identical consumers is born. They live for two subsequent dates. The mass of consumers in every generation is normalized to one. A consumer is described by her consumption set $X = \mathbb{R}^2_{++}$, homothetic utility function $u : X \to \mathbb{R}$ and endowment of labor $\ell = 2$ when young.

Consumers are supposed to satisfy the following assumptions:

(C.1) $u \in C^2(X, \mathbb{R})$ with $Du(c) \in \mathbb{R}_+ \times \mathbb{R}_{++}$ for all c and $v^T D^2 u(c) v < 0$ for all c and $v \neq 0$ with $v \cdot Du(c) = 0$.

(C.2) $u^{-1}(a) = \{c \in X \mid u(c) = a\}$ is closed in $\mathbb{R}^2$ for all a.

The first assumption states that the utility function is twice differentiable, strongly monotone and strictly quasi-concave. The second assumption implies indifference curves do not tend to the axes.

Wages W_t are paid per unit of human capital. Consumers can use labor to accumulate human capital through education and work. Let Z_t be the level of knowledge at date t, and ℓ^e and ℓ^w the units of time used for education and work, respectively. Then the human capital of a consumer is $h_t = \alpha Z_t \ell^e \ell^w$. Therefore consumers choose to spend half of their time on education and half of their time on work $\ell^e = \ell^w = 1$ so $h_t = \alpha Z_t$ for every t. If almost all consumers in generation t use ℓ^e units of time on education, then the level of knowledge at date $t+1$ is $Z_{t+1} = \alpha Z_t \ell^e$. Hence $Z_{t+1} = \alpha Z_t$ and $h_{t+1} = \alpha h_t$ for every t (see Lucas, 1988).

At date t, in addition to educating and working, a consumer can transform the consumption good x_t into capital k_t at date $t+1$ by use of a linear technology $k_t = \beta x_t$ with $\beta > 0$ and rent the capital out to firms at price R_{t+1}. Moreover she can exchange the consumption good for money m_t to transfer purchasing power from date t to $t+1$. Money is a durable good that yields no utility. The real return on money is P_t/P_{t+1} and the real interest rate is $P_t/P_{t+1} - 1$.

There is a continuum of identical firms that transform capital and labor at date t into the consumption good at date t. A firm is described by its constant returns to scale production function $F : \mathbb{R}^2_{++} \to \mathbb{R}_{++}$ with $Y = F(K, L)$. Firms are supposed to satisfy the following assumptions:

(F.1) $F \in C^2(\mathbb{R}^2_{++}, \mathbb{R}_{++})$ with $DF(K,L) \in \mathbb{R}^2_{++}$ for all (K,L) and $v^T D^2 F(K,L) v < 0$ for all (K,L) and $v \neq 0$ with $v \cdot DF(K,L) = 0$.

(F.2) $\lim\limits_{K \to 0} D_K F(K,1) = \lim\limits_{L \to 0} D_L F(1,L) = \infty$ and $\lim\limits_{K \to \infty} D_K F(K,1) = \lim\limits_{L \to \infty} D_L F(1,L) = 0$.

The first assumption states that the production function is twice differentiable, strongly monotone and strictly quasi-concave. The second assumption is the standard Inada conditions.

Our economies have two distinctive features. The first feature is that there are two technologies for producing capital, β_L and β_H, with $\beta_L < \beta_H$. Some consumers, denoted L-consumers, have access to the low-productive technology, and others, denoted H-consumers, have access to the high-productive technology. The proportions of consumers with access to the two technologies are $\delta_L, \delta_H > 0$.

The second feature is that consumers cannot commit to repay loans. Therefore L-consumers are not willing to lend their savings to H-consumers although the latter have access to a superior technology compared with the former. Consequently, consumers save by producing capital on their own or by holding money, but not by lending. From an efficiency point of view L-consumers should lend to H-consumers rather than produce capital themselves.

An economy is a description of consumers and firms $\mathscr{E} = (u, \beta_L, \beta_H, \delta_L, \delta_H, F)$.

The consumer problem

The problem of an i-consumer with $i \in \{L, H\}$ in generation t is

$$\max_{(c^y_{it}, c^o_{it}, m_{it}, x_{it}, k_{it})} \quad u(c^y_{it}, c^o_{it})$$
$$\text{s.t.} \quad \begin{cases} P_t c^y_{it} + m_{it} + P_t x_{it} & \leq & W_t h_t \\ P_{t+1} c^o_{it} & \leq & m_{it} + R_{t+1} k_{it} \\ k_{it} & \leq & \beta_i x_{it} \\ m_{it}, x_{it} & \geq & 0. \end{cases} \tag{1}$$

A consumption plan for an i-consumer in generation t is a consumption bundle, input and output in capital production and nominal savings $(c_{it}, x_{it}, k_{it}, m_{it})$ satisfying the inequalities in Problem (1).

Since the utility function is homothetic implying the demand function is homogeneous of degree one in income it is helpful to consider the following normalized consumer problem

$$\max_{(c^y, c^o)} \quad u(c^y, c^o)$$
$$\text{s.t.} \quad pc^y + c^o \leq p.$$

Assumptions (C.1) and (C.2) imply there is a differentiable function $f : \mathbb{R}_{++} \to X$ such that $f(p) = (f^y(p), f^o(p))$ is a solution to the problem (see Tvede, 2010). The function $s : \mathbb{R}_{++} \to]0,1[$ defined by $s(p) = 1 - f^y(p)$ is related to the function f by $f^y(p) = 1 - s(p)$ and $f^o(p) = ps(p)$. The real interest rate is $p - 1$ and if the income is one when young and zero when old, then the real savings are $s(p)$. Real savings are assumed to be bounded away from zero for high real interest rates:

(C.3) $\liminf_{p\to\infty} s(p) > 0$.

The assumption is satisfied for many utility functions, including Cobb–Douglas and CES utility functions with positive elasticity of substitution.

For convenience consumption is assumed to be positive both when young and when old. Savings $W_t h_t - P_t c_t^y$ are positive for all real interest rates because consumers have no income when old. However, since we use real savings s rather than demand f in our analysis, it is not important whether consumers consume both when young and when old $X = \mathbb{R}^2_{++}$ or solely when old $X = \{0\} \times \mathbb{R}_{++}$. In the latter case real savings are constant and equal to one, making real savings independent of real interest rates.

Let $r_t = R_t/P_t$ and $w_t = W_t/P_t$ be the real price of capital and the real wage rate, respectively, and $p_t = P_{t-1}/P_t$ the relative price for the consumption good. If $r_{t+1}\beta_i < p_{t+1}$ the return on capital production is lower than the return on money and the solution to the consumer problem (1) is

$$\begin{cases} c_{it} &= f(p_{t+1})w_t h_t \\ x_{it} &= 0 \\ k_{it} &= 0 \\ m_{it} &= P_t s(p_{t+1})w_t h_t. \end{cases}$$

If $r_{t+1}\beta_i > p_{t+1}$ the solution is

$$\begin{cases} c_{it} &= f(r_{t+1}\beta_i)w_t h_t \\ x_{it} &= s(r_{t+1}\beta_i)w_t h_t \\ k_{it} &= \beta_i s(r_{t+1}\beta_i)w_t h_t \\ m_{it} &= 0. \end{cases}$$

If $r_{t+1}\beta_i = p_{t+1}$, the return on capital production is equal to the return on money and all convex combinations of the two solutions are solutions.

For $r_{t+1}\beta_L > p_{t+1}$ all consumers save by producing capital. For $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$ L-consumers save by holding money and H-consumers save by producing capital. Finally, for $r_{t+1}\beta_H < p_{t+1}$ all consumers save by holding money.

The firm problem

The problem of a firm at date t is

$$\begin{aligned} \max_{(K_t,L_t,Y_t)} \quad & P_t Y_t - R_t K_t - W_t L_t \\ \text{s.t.} \quad & Y_t \leq F(K_t, L_t). \end{aligned}$$

A production plan for a firm at date t is inputs and output (K_t, L_t, Y_t) satisfying the inequality in the problem of the firm. The marginal products of capital and labor $\eta_K, \eta_L : \mathbb{R}_{++} \to \mathbb{R}_{++}$ are defined by

$$\begin{aligned} \eta_K(K) &= D_K F(K,1) \\ \eta_L(K) &= D_L F(K,1). \end{aligned}$$

(F.1) and (F.2) imply $\eta_L'(K) > 0 > \eta_K'(K)$ for all K, $\lim_{K\to 0} \eta_K(K) = \lim_{K\to\infty} \eta_L(K) = \infty$ and $\lim_{K\to\infty} \eta_K(K) = \lim_{K\to 0} \eta_L(K) = 0$.

The capital share of production is assumed to be bounded away from one for low levels of capital:

(F.3) $\limsup_{K\to 0} \dfrac{D_K F(K,1)K}{F(K,1)} < 1.$

The assumption is satisfied for many production functions, including both Cobb–Douglas and CES production functions.

Profit maximization implies that the solution to the firm's problem satisfies the standard first-order conditions

$$\begin{aligned} r_t &= \eta_K(K_t) \\ w_t &= \eta_L(K_t) \;=\; \eta_L \circ \eta_K^{-1}(r_t). \end{aligned}$$

3 Equilibria

In the present section we define equilibria and steady states and provide results on their existence and dynamic properties.

Definition of equilibrium

Equilibria are sequences of prices, consumption plans and production plans such that consumers maximize their utilities, firms maximize their profits and markets clear.

Definition 1 *An* ***equilibrium*** *is a sequence of prices, consumption plans and production plans*

$$((\bar{P}_t, \bar{R}_t, \bar{W}_t)_{t\in\mathbb{Z}}, (\bar{c}_{it}, \bar{x}_{it}, \bar{k}_{it}, \bar{m}_{it})_{i\in\{L,H\}, t\in\mathbb{Z}}, (\bar{K}_t, \bar{L}_t, \bar{Y}_t)_{t\in\mathbb{Z}}),$$

for which there is a stock of money $M \geq 0$, such that

- *Consumers maximize utility: $(\bar{c}_{it}, \bar{x}_{it}, \bar{k}_{it}, \bar{m}_{it})$ is a solution to the problem of an i-consumer in generation t for both i and every t.*
- *Firms maximize profits: $(\bar{K}_t, \bar{L}_t, \bar{Y}_t)$ is a solution to the problem for a firm at date t for every t.*

- *Markets clear:*

$$\begin{cases} \sum_i \delta_i(\bar{c}^y_{it}+\bar{c}^o_{it-1}+\bar{x}_{it}) & = & \bar{Y}_t \\ \sum_i \delta_i \bar{k}_{it} & = & \bar{K}_{t+1} \\ h_t & = & \bar{L}_t \\ \sum_i \delta_i \bar{m}_{it} & = & M \end{cases}$$

for every t.

A ***real equilibrium*** *is an equilibrium for which $M = 0$, a* ***monetary equilibrium*** *is an equilibrium for which $M > 0$.*

For simplicity we normalize the price of money to one and adjust the stock of money rather than having variable prices for money and a fixed stock of money.

The budget constraints reveal that money allows transfers of resources across generations. On aggregate young consumers have less than their wage available for consumption and production of capital and old consumers have more than their return on capital available for consumption. It is well known that transfers of resources across generations can be welfare improving (Samuelson, 1958; Diamond, 1965; Gale, 1973; Balasko and Shell 1980, 1981; Tirole, 1985).

Let λ_{it}, respectively $(1-\lambda_{it})$, be the share of an i-consumer's savings that goes into production of capital, respectively into holding money. Then equilibria are associated with solutions $(r_t, P_t, \lambda_{Lt}, \lambda_{Ht})_{t\in\mathbb{Z}}$ to the following difference equation:

$$\begin{aligned} \eta_K^{-1}(r_{t+1}) &= \frac{\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L)+\lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)}{\alpha}\eta_L\circ\eta_K^{-1}(r_t) \\ \frac{M}{h_t P_t} &= ((1-\lambda_{Lt})\delta_L+(1-\lambda_{Ht})\delta_H)s\left(\frac{P_t}{P_{t+1}}\right)\eta_L\circ\eta_K^{-1}(r_t) \\ \lambda_{it} &\in \begin{cases} \{0\} & \text{for } r_{t+1} < \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \\ [0,1] & \text{for } r_{t+1} = \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \quad \text{for both } i\text{'s.} \\ \{1\} & \text{for } r_{t+1} > \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \end{cases} \end{aligned} \tag{2}$$

The first equation in Equation (2) is market clearing for the capital market with the capital demanded by firms on the left and the capital supplied by consumers on the right. The first part is share of income allocated to investment in capital production and the second part is the real wage. The second equation is market clearing for the money market with real money supplied by old consumers and real money demanded by young consumers. The first part on the right is the share of income allocated to holding money and the second part is the real wage. Both equations are per unit of human capital. The third equation describes how the share of income invested in capital production is determined for both i's. For fixed r_{t+1} and P_{t+1} the three equations determine r_t, P_t and λ_{it} for both i's.

Real equilibria

An equilibrium is real provided $\lambda_{Lt}, \lambda_{Ht} = 1$ for every t and monetary provided $\lambda_{Lt} + \lambda_{Ht} < 2$ for every t. Therefore real equilibria are associated with solutions $(r_t)_{t \in \mathbb{Z}}$ to the following difference equation

$$\eta_K^{-1}(r_{t+1}) \;=\; \frac{\beta_L \delta_L s(r_{t+1}\beta_L) + \beta_H \delta_H s(r_{t+1}\beta_H)}{\alpha}\, \eta_L \circ \eta_K^{-1}(r_t). \tag{3}$$

Our first result is that real equilibria exist.

Theorem 1 *Consider an economy $\mathscr{E}$. There are real equilibria.*

Proof: Real equilibria are associated with solutions $(r_t)_{t \in \mathbb{Z}}$ to Equation (3). First, for all $r_{t+1} > 0$ there is a unique $r_t > 0$ such that (r_t, r_{t+1}) satisfies Equation (3). Second, since $s(p) < 1$ for all $p > 0$, there is $r_{t+1} > 0$ such that

$$\eta_K^{-1}(r_{t+1}) \;>\; \frac{\beta_L \delta_L s(r_{t+1}\beta_L) + \beta_H \delta_H s(r_{t+1}\beta_H)}{\alpha}\, \eta_L \circ \eta_K^{-1}(r_t).$$

Since (C.3) implies there are $\bar{p}, \bar{s} > 0$ such that $p \geq \bar{p}$ implies $s(p) \geq \bar{s}$, there is $r_{t+1} > 0$ such that

$$\eta_K^{-1}(r_{t+1}) \;<\; \frac{\beta_L \delta_L s(r_{t+1}\beta_L) + \beta_H \delta_H s(r_{t+1}\beta_H)}{\alpha}\, \eta_L \circ \eta_K^{-1}(r_t).$$

By continuity for all $r_t > 0$ there is $r_{t+1} > 0$ such that (r_t, r_{t+1}) satisfies Equation (3) difference equation. Therefore for all $\bar{r} > 0$ there is $(r_t)_{t \in \mathbb{Z}}$ with $r_0 = \bar{r}$ that is a solution to the difference equation. □

Consumption in real equilibria is not weakly Pareto optimal (see Balasko and Shell, 1980) because L-consumers and H-consumers in the same generation have different marginal rates of substitution. Every consumer saves by producing capital, but L-consumers get a lower return than H-consumers. Therefore transfers of resources from L-consumers to H-consumers when young and from H-consumers to L-consumers when old could improve welfare for everybody.

Steady states

Steady states are equilibria for which consumption possibilities per unit of human capital are identical across generations.

Definition 2 *A **steady state** is an equilibrium*

$$((\bar{P}_t,\bar{R}_t,\bar{W}_t)_{t\in\mathbb{Z}},(\bar{c}_{it},x_{it},\bar{k}_{it},\bar{m}_{it})_{i\in\{L,H\},t\in\mathbb{Z}},(\bar{K}_t,\bar{L}_t,\bar{Y}_t)_{t\in\mathbb{Z}})$$

for which there are $\bar{p},\bar{r},\bar{w}>0$ such that $(\bar{p}_t,\bar{r}_t,\bar{w}_t)=(\bar{p},\bar{r},\bar{w})$ for every t and $(\bar{c}_{it},\bar{x}_{it},\bar{k}_{it})=\alpha^t(\bar{c}_{i0},\bar{x}_{0i},\bar{k}_{i0})$ for both i and every t and $(\bar{K}_t,\bar{L}_t,\bar{Y}_t)=\alpha^t(\bar{K}_0,\bar{L}_0,\bar{Y}_0)$ for every t.

Our second result is that all economies have real steady states and that all economies, for which savings are strictly larger than demand for capital for the real interest rate $r=\alpha/\beta_L$, have monetary steady states.

Theorem 2 *Consider an economy $\mathscr{E}$.*

- *There are real steady states.*
- *There are multiple real steady states provided there are $r'>r>0$ such that,*

$$\begin{aligned}\eta_K^{-1}(r) &< \frac{\beta_L\delta_L s(r\beta_L)+\beta_H\delta_H s(r\beta_H)}{\alpha}\,\eta_L\circ\eta_K^{-1}(r)\\ \eta_K^{-1}(r') &> \frac{\beta_L\delta_L s(r'\beta_L)+\beta_H\delta_H s(r'\beta_H)}{\alpha}\,\eta_L\circ\eta_K^{-1}(r').\end{aligned}$$

- *There are monetary steady states provided that for $r=\alpha/\beta_L$,*

$$\eta_K^{-1}(r) < \frac{\beta_L\delta_L s(r\beta_L)+\beta_H\delta_H s(r\beta_H)}{\alpha}\,\eta_L\circ\eta_K^{-1}(r).$$

Proof: "Existence of real steady states" Real equilibria are associated with solutions $(r_t)_{t\in\mathbb{Z}}$ to Equation (3) or equivalently solutions $(K_t)_{t\in\mathbb{Z}}$ to the following difference equation:

$$\frac{\alpha}{\beta_L\delta_L s(\eta_K(K_{t+1})\beta_L)+\beta_H\delta_H s(\eta_K(K_{t+1})\beta_H)}K_{t+1} \;=\; F(K_t,1)-D_KF(K_t,1)K_t. \qquad (4)$$

Since

$$F(K_t,1)-D_KF(K_t,1)K_t \;=\; D_LF(K_t,1) \;=\; D_LF(1,1/K_t),$$

$K_t\to 0$ implies $K_{t+1}\to 0$ and $K_t\to\infty$ implies $K_{t+1}\to\infty$. To evaluate K_{t+1}/K_t for (K_t,K_{t+1}) satisfying Equation (4) it is useful to rewrite Equation (4):

$$\frac{\alpha}{\beta_L\delta_L s(\eta_K(K_{t+1})\beta_L)+\beta_H\delta_H s(\eta_K(K_{t+1})\beta_H)}\frac{K_{t+1}}{K_t} \;=\; \left(1-\frac{D_KF(K_t,1)K_t}{F(K_t,1)}\right)\frac{F(K_t,1)}{K_t}.$$

(F.1)–(F.3) imply

$$
\begin{aligned}
\liminf_{K_t \to 0} \left(1 - \frac{D_K F(K_t,1) K_t}{F(K_t,1)}\right) \frac{F(K_t,1)}{K_t} &= \infty \\
\limsup_{K_t \to \infty} \left(1 - \frac{D_K F(K_t,1) K_t}{F(K_t,1)}\right) \frac{F(K_t,1)}{K_t} &= 0.
\end{aligned}
$$

(C.3) implies that if (K_t, K_{t+1}) satisfies Equation (4), then

$$
\begin{aligned}
\lim_{K_t \to 0} \frac{K_{t+1}}{K_t} &= \infty \\
\lim_{K_t \to \infty} \frac{K_{t+1}}{K_t} &= 0.
\end{aligned}
$$

Since F and s are continuous functions, there is K_t such that (K_t, K_{t+1}) with $K_{t+1} = K_t$ satisfies Equation (4).

"Existence of multiple real steady states" Consider the function $\Gamma : \mathbb{R}_{++} \to \mathbb{R}$ defined by

$$
\Gamma(r) = \eta_K^{-1}(r) - \frac{\beta_L \delta_L s(r\beta_L) + \beta_H \delta_H s(r\beta_H)}{\alpha} \eta_L \circ \eta_K^{-1}(r).
$$

Then there is $\varepsilon > 0$ such that $r < \varepsilon$ implies $\Gamma(r) > 0$ and $r > 1/\varepsilon$ implies $\Gamma(r) < 0$ according to the proof for existence of real steady states. Moreover $\Gamma(\bar{r}) < 0 < \Gamma(\bar{r}')$. Hence there are at least three real steady states.

"Existence of monetary steady states" In monetary steady states $\bar{p} = \alpha$ because otherwise $M/h_t\bar{P}_t$ is not constant across dates. Therefore $\alpha/\beta_H \leq \bar{r} \leq \alpha/\beta_L$ at monetary steady states, for otherwise either nobody would hold money or nobody would produce capital. It suffices to find the steady state solution $(r, P, \lambda_L, \lambda_H)$ to the first difference equation in (2) with $\lambda_H + \lambda_L \in (0,2)$. Given this solution, the second difference equation in (2) implicitly defines an $M > 0$ for which the economy is in a monetary equilibrium.

Suppose that for $r = \alpha/\beta_L$,

$$
\frac{\alpha}{\beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) \geq \eta_L \circ \eta_K^{-1}(r).
$$

Then the condition in Theorem 2 implies that there is $\lambda_L \in]0,1]$ such that

$$
\frac{\alpha}{\lambda_L \beta_L \delta_L s(r\beta_L) + \beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) = \eta_L \circ \eta_K^{-1}(r).
$$

Hence there is a monetary steady state with $\bar{r} = \alpha/\beta_L$ where part of the L-consumers saves by holding money and the rest saves by producing capital.

Suppose that for $r = \alpha/\beta_L$ and $r' = \alpha/\beta_H$,

$$\frac{\alpha}{\beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) \; < \; \eta_L \circ \eta_K^{-1}(r)$$
$$\frac{\alpha}{\beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') \; > \; \eta_L \circ \eta_K^{-1}(r').$$

Then there is $r'' \in]\alpha/\beta_L, \alpha/\beta_H[$ such that

$$\frac{\alpha}{\beta_H \delta_H s(r''\beta_H)} \eta_K^{-1}(r'') \; = \; \eta_L \circ \eta_K^{-1}(r'').$$

Hence, by continuity, there is a monetary steady state with $\bar{r} = r''$ where all L-consumers save by holding money and all H-consumers save by producing capital.

Suppose that for $r = \alpha/\beta_L$ and $r' = \alpha/\beta_H$,

$$\frac{\alpha}{\beta_H \delta_H s(r\beta_H)} \eta_K^{-1}(r) \; < \; \eta_L \circ \eta_K^{-1}(r)$$
$$\frac{\alpha}{\beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') \; \leq \; \eta_L \circ \eta_K^{-1}(r').$$

Then there is $\lambda_H \in]0, 1]$ such that

$$\frac{\alpha}{\lambda_H \beta_H \delta_H s(r'\beta_H)} \eta_K^{-1}(r') \; = \; \eta_L \circ \eta_K^{-1}(r').$$

Hence the condition in Theorem 2 implies there is a monetary steady state with $\bar{r} = r'$ where part of the H-consumers saves by producing capital and the rest saves by holding money. □

Real steady states exist, but need not be unique, and monetary steady states can exist. There are three types of monetary steady states: L-consumers are indifferent between saving by holding money and saving by producing capital $\bar{p} = \beta_L \bar{r}$; L-consumers prefer to save by holding money and H-consumers prefer to save by producing capital $\bar{p} \in]\beta_L \bar{r}, \beta_H \bar{r}[$; and, finally, H-consumers are indifferent between saving by holding money and saving by producing capital $\bar{p} = \beta_H \bar{r}$.

Consumption in the two first types of monetary steady states is not weakly Pareto optimal because L-consumers and H-consumers in the same generation have different marginal rates of substitution. Again, transfers of resources from L-consumers to H-consumers when young and from H-consumers to L-consumers when old could improve welfare for everybody. Consumption in the last type of monetary steady states is weakly Pareto optimal because L-consumers and H-consumers in the same generation have identical marginal rates of substitution.

Money crowds out capital accumulation in the sense that the minimal and maximal capital stocks in monetary steady states are lower than the minimal and maximal capital stocks of capital in real steady states. In real steady states the capital stock can be lower than, equal to or higher than the golden rule capital stock. Therefore crowding out of capital can decrease as well as increase welfare.

4 Dynamics

Equilibria can be everything from steady states to chaotic. However complicated dynamics are eliminated by the assumption that savings are non-decreasing in the real interest rate.

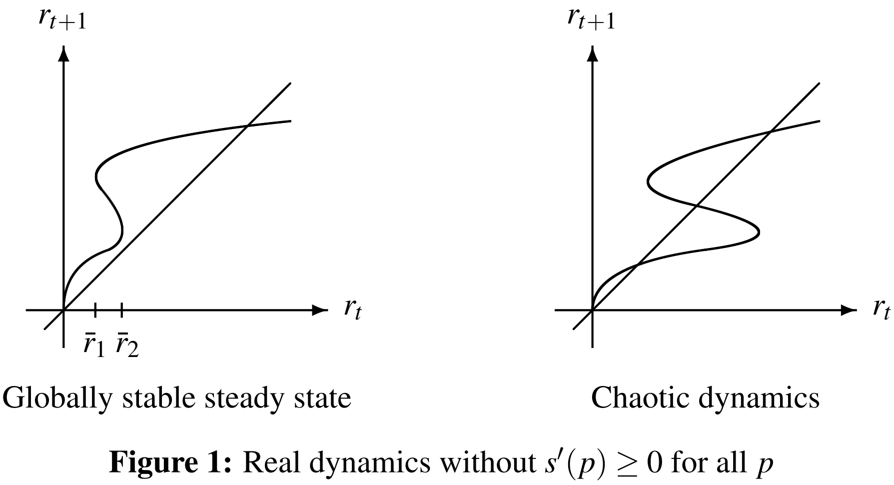
Real dynamics

Equation (3) describing the dynamics of real equilibria is equivalent to

$$\frac{\alpha}{\beta_L\delta_L s(r_{t+1}\beta_L)+\beta_H\delta_H s(r_{t+1}\beta_H)}\,\eta_K^{-1}(r_{t+1}) \;=\; \eta_L\circ\eta_K^{-1}(r_t).$$

The first part on the left need not be monotonic in r_{t+1}, the second part on the left is decreasing in r_{t+1} and the part on the right is decreasing in r_t. Since the range of $\eta_L\circ\eta_K^{-1}$ is $]0,\infty[$ and the part on the right is decreasing in r_t, for all r_{t+1} there is a unique r_t such that (r_t, r_{t+1}) is a solution to Equation (3). Consequently the backward equilibrium dynamics of real equilibria are well defined.

If savings are not non-decreasing in the real interest rate, then the first part on the left can be non-monotonic in r_{t+1}. Consequently the forward equilibrium dynamics need not be well defined as illustrated in Figure 1. In the left panel there is a unique steady state and it is globally stable even though the forward equilibrium dynamics are not well defined for r_t between $\bar{r}_1$ and $\bar{r}_2$. In the right panel there are three steady states and they locally stable, but there are cycles of every period around the second steady state. If savings are



Globally stable steady state　　　　Chaotic dynamics

Figure 1: Real dynamics without $s'(p) \geq 0$ for all p

non-decreasing in the real rate of interest, then the first part on the left is non-increasing in r_{t+1} implying the part on the left is decreasing in r_{t+1}. Since the range of η_K^{-1} is $]0,\infty[$ and the part on the left is decreasing in r_{t+1}, for all r_t there is a unique r_{t+1} such that (r_t, r_{t+1}) is a solution to Equation (3). Consequently the forward equilibrium dynamics are well defined. All in all, if savings are non-decreasing in the real interest rate, then for all $r > 0$ there is a unique real equilibrium with $r_0 = r$.

Since the assumption that savings are non-decreasing in the real interest rate implies $r_s > r_t$ if and only if $r_{s+1} > r_{t+1}$ for real equilibria, there are three types of real equilibria: increasing real interest rate $r_t < r_{t+1}$ for every t; constant real interest rate $r_t = r_{t+1}$ for every t; and, decreasing real interest rate $r_t > r_{t+1}$ for every t. Therefore there are no fluctuations provided savings are non-decreasing in the real interest rate. However there can be multiple steady states. If real steady states are locally isolated as in Figure 2, then their stability properties alternate between being locally stable as $\bar{r}_1$ and $\bar{r}_3$ and being locally unstable as $\bar{r}_2$ in the left panel. If the real steady state is unique, it is globally stable as $\bar{r}$ in the right

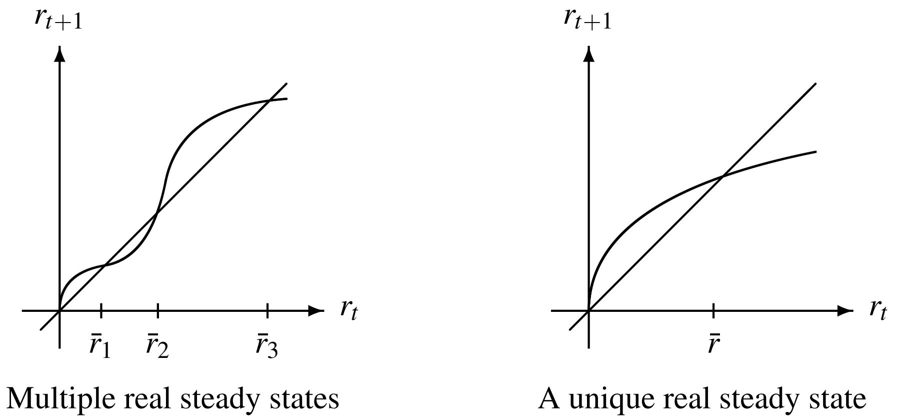


Figure 2: Real dynamics with $s'(p) \geq 0$ for all p

panel. In the proof of Theorem 2 it is shown for (r_t, r_{t+1}) being a solution to Equation (3) that if r_t converge to zero, then $r_t < r_{t+1}$, and if r_t tends to infinity, then $r_t > r_{t+1}$. The first type converges to a steady state in the forward dynamics and either zero as for $r_t < \bar{r}_1$ in left panel or another steady state in the backward dynamics as for $r_t \in]\bar{r}_2, \bar{r}_3[$ in the left panel. The third type converges to a steady state in the forward dynamics and either infinity as for $r_t > \bar{r}_3$ in the left panel or another steady state as for $r_t \in]\bar{r}_1, \bar{r}_2[$ in the backward dynamics.

Growth rates are identical in all steady states because the source of growth is accumulation of human capital and accumulation of human capital is independent of the evolution of the economy. However different real steady states are associated with different capital stocks per unit of human capital, where the stock of capital is the smaller the higher the real return on capital is. If an economy is at a real steady state with a small capital stock like $\eta_K^{-1}(\bar{r}_3)$, then the capital stock needs to be pushed to a capital stock above $\eta_K^{-1}(\bar{r}_2)$ in order to make the economy converge to a high capital stock like $\eta_K^{-1}(\bar{r}_1)$. For a small push to a capital stock between $\eta_K^{-1}(\bar{r}_3)$ and $\eta_K^{-1}(\bar{r}_2)$ the capital stock converges to $\eta_K^{-1}(\bar{r}_3)$ as t tends to infinity.

Monetary dynamics

Monetary dynamics are more delicate. The reason is that the demand for money depends on whether: L-consumers are indifferent between holding money and producing capital and H-consumers prefer to produce capital; L-consumers prefer to hold money and H-consumers prefer to produce capital; and, L-consumers prefer to hold money and H-consumers are indifferent between holding money and producing capital.

In Equation (2), which defines the dynamics of equilibria, it is convenient to replace P_t with Q_t where $Q_t = \alpha^t P_t$ for every t, so $p_{t+1} = P_t/P_{t+1} = (1/\alpha)Q_t/Q_{t+1}$. In case L-consumers are indifferent between holding money and producing capital and H-consumers prefer to produce capital $r_{t+1}\beta_L = p_{t+1}$, Equation (2) without the last equation is equivalent to

$$\begin{aligned}\frac{\alpha}{\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L)+\beta_H\delta_H s(r_{t+1}\beta_H)}\eta_K^{-1}(r_{t+1}) &= \eta_L\circ\eta_K^{-1}(r_t)\\ (1-\lambda_{Lt})\delta_L s(r_{t+1}\beta_L) &= \frac{1}{\eta_L\circ\eta_K^{-1}(r_t)}\frac{M}{h_0 Q_t}.\end{aligned}$$

For fixed r_t and Q_t the two equations determine r_{t+1} and λ_{Lt} while Q_{t+1} is determined by $Q_{t+1} = (1/(\alpha\beta_L))Q_t/r_{t+1}$. However, since λ_{Lt-1} has no impact on r_{t+1}, λ_{Lt} and Q_{t+1}, the dynamics are two-dimensional. In a steady state $\bar{p} = 1/\alpha$ so $\bar{r} = 1/(\alpha\beta_L)$. Therefore there is at most one monetary steady state where L-consumers are indifferent between holding money and producing capital and H-consumers prefer to produce capital independent of whether s is non-decreasing. Concerning the forward dynamics there is at most one r_{t+1}, λ_{Lt} and Q_{t+1} for fixed r_t and Q_t provided s is non-decreasing.

In case L-consumers prefer to hold money and H-consumers prefer to produce capital $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$ Equation (2) without the last equation is equivalent to

$$\begin{aligned}\frac{\alpha}{\beta_H\delta_H s(r_{t+1}\beta_H)}\eta_K^{-1}(r_{t+1}) &= \eta_L\circ\eta_K^{-1}(r_t)\\ \delta_L s\left(\frac{1}{\alpha}\frac{Q_t}{Q_{t+1}}\right) &= \frac{1}{\eta_L\circ\eta_K^{-1}(r_t)}\frac{M}{h_0 Q_t}.\end{aligned}$$

The first equation defines the evolution of r_t. For a given evolution of r_t the second equation defines the evolution of Q_t. Hence the dynamics are one-dimensional. Qualitatively the relation between r_t and r_{t+1} is similar to the relation between them in real equilibria defined implicitly in Equation (3). The relation between r_t and r_{t+1} is increasing provided s is non-decreasing and in that case: forward as well as backward dynamics are well defined; and monetary steady states alternate between being locally stable and being locally unstable provided they are locally isolated.

In case L-consumers prefer to hold money and H-consumers are indifferent between holding money and producing in capital $r_{t+1}\beta_H = p_{t+1}$ Equation (2) without the last equation is equivalent to

$$\begin{aligned}\frac{\alpha}{\lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)}\eta_K^{-1}(r_{t+1}) &= \eta_L\circ\eta_K^{-1}(r_t)\\ \delta_L s(r_{t+1}\beta_L)+(1-\lambda_{Ht})\delta_H s(r_{t+1}\beta_H) &= \frac{1}{\eta_L\circ\eta_K^{-1}(r_t)}\frac{M}{h_0 Q_t}.\end{aligned}$$

For fixed r_t and Q_t the two equations determine r_{t+1} and λ_{Ht} while Q_{t+1} is determined by $Q_{t+1} = (1/(\alpha\beta_H))Q_t/r_{t+1}$. However, since λ_{Lt-1} has no impact on r_{t+1}, λ_{Lt} and Q_{t+1}, the dynamics are two-dimensional. In a steady state $\bar{p} = 1/\alpha$ so $\bar{r} = 1/(\alpha\beta_H)$. Therefore there is at most one monetary steady state where L-consumers prefer to hold money and and H-consumers are indifferent between holding money and investing in capital. Concerning the forward dynamics there is at most one r_{t+1}, λ_{Lt} and Q_{t+1} for fixed r_t and Q_t provided s is non-decreasing.

In all three cases the first equation can be used to find the return on capital part of equilibria $(r_t)_{t\in\mathbb{Z}}$ and the second equation can be used to find the price level part of equilibria $(Q_t)_{t\in\mathbb{Z}}$. However they need to be compatible in order to be part of an equilibrium: $r_{t+1} = (\beta_L/\alpha)Q_t/Q_{t+1}$ in case L-consumers are indifferent between holding money and producing capital; $r_{t+1}\beta_L < (\beta_L/\alpha)Q_t/Q_{t+1} < r_{t+1}\beta_H$ in case L-consumers prefer to hold money and H-consumers prefer to produce capital; and, $r_{t+1} = (\beta_H/\alpha)Q_t/Q_{t+1}$ in case H consumers are indifferent between holding money and producing capital. What makes monetary dynamics more delicate than real dynamics are that equilibria can switch between the different cases between dates and that λ's can switch between dates in the first and the third case.

5 Printing money

In the present section we study how bona fide monetary policies in form of printing money and distributing it among consumers influence steady states and dynamics.

Steady states with money printing

Let N_{it}^y (N_{it}^o) be the monetary transfers to young (old) i-consumers at date t. For given prices monetary transfers increase incomes of consumers receiving them and these income increases lead to higher consumption both when young and when old provided consumption when young and consumption when old are normal goods. Therefore for given prices monetary transfers to young consumers, but not old consumers, lead to higher savings so monetary transfers to young H-consumers lead to more production of capital.

Three scenarios are considered, namely newly printed money is distributed: exclusively to young H-consumers; to both kinds of young consumers in equal amounts; and, exclusively to young L-consumers. In the first and third scenarios monetary transfers can depend on the kind of as well as the age of the consumers and in the second scenario monetary transfers can depend solely on the age of the consumers. In the first scenario $N^y_{Ht} \geq 0$ and $N^y_{Lt} = N^o_{Ht} = N^o_{Lt} = 0$ for every t; in the second $N^y_{Ht} = N^y_{Lt} \geq 0$ and $N^o_{Ht} = N^o_{Lt} = 0$ for every t; and, in the third $N^y_{Lt} \geq 0$ and $N^y_{Ht} = N^0_{Ht} = N^o_{Lt} = 0$ for every t.

Monetary equilibria are associated with solutions $(r_t, P_t, \lambda_{Lt}, \lambda_{Ht})_{t \in \mathbb{Z}}$ to the following difference equation:

$$
\begin{aligned}
\eta_K^{-1}(r_{t+1}) - \frac{\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L)}{\alpha}\frac{N^y_{Lt}}{h_tP_t} - \frac{\lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)}{\alpha}\frac{N^y_{Ht}}{h_tP_t} \\
= \frac{\lambda_{Lt}\beta_L\delta_L s(r_{t+1}\beta_L) + \lambda_{Ht}\beta_H\delta_H s(r_{t+1}\beta_H)}{\alpha}\eta_L \circ \eta_K^{-1}(r_t) \\
\frac{M_t}{h_tP_t} - (1-\lambda_{Lt})\delta_L s\left(\frac{P_t}{P_{t+1}}\right)\frac{N^y_{Lt}}{h_tP_t} - (1-\lambda_{Ht})\delta_H s\left(\frac{P_t}{P_{t+1}}\right)\frac{N^y_{Ht}}{h_tP_t} \\
= ((1-\lambda_{Lt})\delta_L + (1-\lambda_{Ht})\delta_H)s\left(\frac{P_t}{P_{t+1}}\right)\eta_L \circ \eta_K^{-1}(r_t) \\
\lambda_{it} \in \begin{cases} \{0\} & \text{for } r_{t+1} < \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \\ [0,1] & \text{for } r_{t+1} = \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \\ \{1\} & \text{for } r_{t+1} > \frac{P_t}{P_{t+1}}\frac{1}{\beta_i} \end{cases} \quad \text{for both } i\text{'s} \\
M_{t+1} = M_t + \delta_L N^y_{Lt+1} + \delta_H N^y_{Ht+1}.
\end{aligned}
\tag{5}
$$

The first equation in Equation (5) is market clearing for the capital market with demand for capital of firms minus supply of capital financed by money transfers on the left and supply of capital financed by wage income on the right. The second equation is market clearing for the money market with supply of money minus demand for money financed by money transfers on the left and demand for money financed by wage income on the right. Both equations are per unit of human capital. The third equation describes how the share of income invested in capital production is determined for both i's. The fourth equation describes how the stock of money evolves.

Our third result is on existence of steady states with monetary transfers to young consumers.

Theorem 3 *Consider an economy $\mathscr{E}$. Let $(\delta_L N^y_{Lt} + \delta_H N^y_{Ht})/M_t = n$ for every t and let $r : \mathbb{R} \to \mathbb{R}$ be defined by $r(n) = \alpha(1-n)/\beta_L$. Then there are monetary steady states with money printing $n \geq 0$ provided that for $r = r(n)$,*

$$\eta_K^{-1}(r) \;<\; \frac{\beta_L \delta_L s(r\beta_L) + \beta_H \delta_H s(r\beta_H)}{\alpha} \eta_L \circ \eta_K^{-1}(r).$$

Proof: In monetary steady states with money printing $\bar{p} = \alpha(1-n)$ because otherwise $M_t/h_t\bar{P}_t$ is not constant across dates. Therefore $\bar{p}/\beta_H \leq \bar{r} \leq \bar{p}/\beta_L$ at monetary steady states with money printing. The proof follows the part on existence of monetary steady states of Theorem 2. □

Theorem 3 generalizes the part on existence of monetary steady states in Theorem 2 to money printing $n \geq 0$. Again, there are three types of monetary steady states:

- $\bar{r} = \bar{p}/\beta_L$, so all L-consumers are indifferent between holding money and producing capital and all H-consumers prefer to save by producing capital.
- $\bar{p}/\beta_H < \bar{r} < \bar{p}/\beta_L$ so all L-consumers prefer to save by holding money and all H-consumers prefer to save by producing capital.
- $\bar{r} = \bar{p}/\beta_H$ so all L-consumers prefer to save by holding money and all H-consumers are indifferent between holding money and producing capital.

For simplicity we assume that: savings are non-decreasing in returns on savings; and, the return on capital production for L-consumers is lower than the return on money and the return on money is lower than the return on capital production for H consumers. The second part implies that L-consumers save by holding money and H-consumers save by producing capital. The first equation in Equation (5) reveals that transfers to young H-consumers drive down the return on capital that the minimal (maximal) return on capital decreases for the first and the second scenarios provided savings s increase in p. Consequently the minimal and the maximal outputs are increased in the first and the second scenarios, but unchanged in the third scenario. For a fixed amount of newly printed money, the larger the transfers to young H-consumers, the more output increases.

At monetary steady states with money printing, transfers to consumers have a direct effect on the return of money. Since the real value of money per unit of human capital has to be constant across dates, the price of the consumption good has to increase faster with monetary transfers than without monetary transfers. Hence printing money decreases the return on existing money.

All in all distributing newly printed money solely to young H-consumers or to both kinds of young consumers increases output by reducing the inefficiencies in capital production, and, distributing money solely to young L-consumers does not change output. Output increasing monetary policies are potentially welfare improving. But redistribution of goods may be needed to ensure that all consumers benefit from the increase in output.

Implementation of social optima

The welfare effects of printing money and transferring them to young H-consumers are difficult to evaluate. On the one hand income is higher because the stock of capital is higher and because of the monetary transfer to H-consumers. On the other hand the return on savings is lower. Therefore a natural question is: Can active bona fide monetary policy achieve an outcome that is superior to any other? To answer this question, suppose that any monetary transfers are possible. Let $T = (N^y_{it}, N^o_{it})_{i\in\{L,H\},t\in\mathbb{Z}}$ be a transfer policy with $(N^y_{it}, N^o_{it+1}) \in \mathbb{R}^2$. If $N^y_{it} > 0$, a consumer receives money at t when young and if $N^y_{it} < 0$ she has to give up money. Similarly, if $N^o_{it} > 0$, a consumer receives money when old at t and if $N^o_{it} < 0$ she has to give up money. The solution to the consumer problem (1) does not change except that for i-consumers in generation t income when young is $W_t h_t + N^y_{it}$ and income when old is N^o_{it+1}.

Whether sequences of consumption plans and production plans are socially optimal depends on the notion of a social optimum. It could be Pareto optimality, some refinement of Pareto optimality such as discounted and weighted utility or perhaps something else. We avoid committing to a specific notion of social optimum by considering a large set of allocations and showing that all allocations in that set can be implemented. Neither the allocations nor the policies are restricted to be stationary.

Definition 3 *An* ***implementable allocation*** *is sequences of consumption plans and production plans* $((c_{it}, x_{it}, k_{it})_{i\in\{L,H\},t\in\mathbb{Z}}, (K_t, L_t, Y_t)_{t\in\mathbb{Z}})$ *with* $x_{Ht} > 0$ *for every* t *for which:*

- *There is a sequence of real returns on capital $(r_t)_{t\in\mathbb{Z}}$ such that for every t,*

$$\begin{cases} \dfrac{D_y u(c^y_{Ht}, c^o_{Ht})}{D_o u(c^y_{Ht}, c^o_{Ht})} &= \beta_H r_{t+1} \\[2ex] \dfrac{D_y u(c^y_{Lt}, c^o_{Lt})}{D_o u(c^y_{Lt}, c^o_{Lt})} &\in [\beta_L r_{t+1}, \beta_H r_{t+1}] \ \textit{for } x_{Lt} = 0 \\[2ex] \dfrac{D_y u(c^y_{Lt}, c^o_{Lt})}{D_o u(c^y_{Lt}, c^o_{Lt})} &= \beta_L r_{t+1} \ \textit{for } x_{Lt} > 0 \\[2ex] D_k F(K_t, h_t) &= r_t. \end{cases}$$

- *Consumption plans and production plans balance such that for every t,*

$$\begin{cases} \sum_i \delta_i (c^y_{it} + c^o_{it-1} + x_{it}) &= Y_t \\ \beta_i x_{it} &= k_{it+1} \ \textit{for both } i \\ \sum_i \delta_i k_{it} &= K_{t+1} \\ h_t &= L_t. \end{cases}$$

The first requirement for an allocation to be implementable is that there are real returns on capital that corresponds to the real returns on capital production for H-consumers and L-consumers and the marginal products of capital in firms. The second requirement is that the allocation is feasible. The set of possible allocations include allocations for which L-consumers and H-consumers in the same generation have identical marginal rates of substitution. These allocations are weakly Pareto optimal so there is no Pareto improvement involving changes at finitely many dates. All other allocations are not weakly Pareto optimal.

Theorem 4 *For all implementable allocations $((c_{it}, x_{it}, k_{it})_{i\in\{L,H\},t\in\mathbb{Z}}, (K_t, L_t, Y_t)_{t\in\mathbb{Z}})$ there are a sequence of prices $(\bar{P}_t, \bar{R}_t, \bar{W}_t)_{t\in\mathbb{Z}}$ and a transfer policy T such that*

$$((\bar{P}_t, \bar{R}_t, \bar{W}_t)_{t\in\mathbb{Z}}, (c_{it}, x_{it}, k_{it})_{i\in\{L,H\},t\in\mathbb{Z}}, (K_t, L_t, Y_t)_{t\in\mathbb{Z}})$$

is an equilibrium with transfers.

Proof: The proof follows the last part of the proof of Theorem 1 in Ghiglino and Tvede (2000). For prices, let: $(\bar{P}_t)_{t\in\mathbb{Z}}$ be defined by $\bar{P}_0 = 1$ and $\bar{P}_{t+1} = \bar{P}_t D_y u(c^y_{Lt}, c^o_{Lt})/D_o u(c^y_{Lt}, c^o_{Lt})$ for every $t \geq 1$ and $\bar{P}_t = \bar{P}_{t+1} D_o u(c^y_{Lt}, c^o_{Lt})/D_y u(c^y_{Lt}, c^o_{Lt})$ for every $t \leq -1$; $(\bar{R}_t)_{t\in\mathbb{Z}}$ by $\bar{R}_t = \bar{P}_t/\beta_L$; and $(\bar{W}_t)_{t\in\mathbb{Z}}$ by $\bar{W}_t = \bar{P}_t D_K F(\delta_L k_{Lt-1} + \delta_H k_{Ht-1}, h_t)$ for every t. For transfers, let: $N_{it}^{y} = \bar{P}_t c_{it}^{y} + \bar{P}_t x_{it} - \bar{W}_t h_t$ and $N_{it}^{o} = \bar{P}_t c_{it-1}^{o} - \bar{R}_t k_{it-1}$ for both i. Then consumers choose $(c_{it}, x_{it}, k_{it})_{i \in \{L,H\}, t \in \mathbb{Z}}$ and firms choose $(K_t, L_t, Y_t)_{t \in \mathbb{Z}}$. Trivially total transfer at every date is zero $\sum_i \delta_i (N_{it}^{y} + N_{it}^{o}) = 0$ for every t. □

To implement allocations that are the outcome of some social planner coming alive at some specific date t the planner would need to tax away the stock of money at that date after trades are carried out.

6 Example

We now turn to an example to further illustrate the mechanics and the welfare implications of monetary policy. This includes a discussion of the boundaries for monetary policy.

Setup

The example we consider is the model presented in Martin and Ventura (2012). It makes two specifications to the general setup. First, consumers care only about consumption when old, i.e. $X = \{0\} \times \mathbb{R}_{++}$. Therefore all income received when young will be saved, i.e. $s(p) = 1$ for all p. Second, production of goods is specified as a Cobb-Douglas function

$$AK_t^{\sigma} L_t^{1-\sigma}$$

with $\sigma \in (0,1)$ being the output elasticity of capital.

For convenience, transfers are written as multiples $\gamma \in \mathbb{R}$ of total wages when young, so $N_{it}^{y} = \gamma_{it}^{y} W_t h_t$ and $N_{it+1}^{o} = \gamma_{it+1}^{o} W_t h_t$. If $\gamma > 0$, a consumer receives money and if $\gamma < 0$ she pays money. The consumer problem at date t for a young consumer is

$$\max_{(c_{it+1}, m_{it}, x_{it}, k_{it})} \quad c_{it+1}$$
$$\text{s.t.} \begin{cases} P_t x_{it} + m_{it} & \leq & W_t h_t + N_{it}^{y} \\ P_{t+1} c_{it+1} & \leq & m_{it} + R_{t+1} k_{it} + N_{it+1}^{o} \\ k_{it} & \leq & \beta_i x_{it} \\ m_{it}, x_{it} & \geq & 0. \end{cases} \tag{6}$$

Since $s(p) = 1$ so $f^y(p) = 0$ and $f^o(p) = p$, the solution to the consumer problem for $r_{t+1}\beta_i < p_{t+1}$ is

$$\begin{cases} c_{it+1} &= p_{t+1}(1+\gamma_{it}^y+\gamma_{it+1}^o)w_t h_t \\ x_{it} &= 0 \\ k_{it} &= 0 \\ m_{it} &= P_t w_t h_t. \end{cases}$$

For $r_{t+1}\beta_i > p_{t+1}$ the solution is

$$\begin{cases} c_{it+1} &= (r_{t+1}\beta_i(1+\gamma_{it}^y)+p_{t+1}\gamma_{it+1}^o)w_t h_t \\ x_{it} &= (1+\gamma_{it}^y)w_t h_t \\ k_{it} &= \beta_i(1+\gamma_{it}^y)w_t h_t \\ m_{it} &= 0. \end{cases}$$

For $r_{t+1}\beta_i = p_{t+1}$, the return on capital production is equal to the return on money and all convex combinations of the two solutions are solutions.

The firm problem at date t is

$$\begin{aligned} \max_{(K_t,L_t,Y_t)} \quad & Y_t - r_t K_t - w_t L_t h_t \\ \text{s.t.} \quad & Y_t \leq AK_t^{\sigma}(L_t h_t)^{1-\sigma}. \end{aligned}$$

For $\kappa = K/(Lh)$ being the capital-labor ratio, the solution satisfies

$$\begin{aligned} r_t &= \eta_K(K_t) &= \sigma A\kappa_t^{\sigma-1}, \\ w_t &= \eta_L(K_t) &= (1-\sigma)A\kappa_t^{\sigma}. \end{aligned}$$

Finally, the path for the aggregate money stock, $\{M_t\}_{t\in\mathbb{Z}}$ is given by

$$M_t = M_{t-1} + \sum_i \delta_i(N_{it}^o+N_{it}^y).$$

Equilibrium

The existence of real equilibria and real steady states is ensured by Theorems 1 and 2. Hence, we proceed with describing the properties of such equilibria. Then we provide conditions for the existence of monetary steady states. These conditions are interesting in themselves because they reveal the boundaries for monetary policy.

For real equilibria assume $M_t = 0$ and $\gamma_{it}^o = \gamma_{it}^y = 0$ for both i and every t. Let $\bar{\beta} = \delta_H \beta_H + \delta_L \beta_L$ be the average productivity in capital production. Equation (3) becomes

$$\kappa_{t+1} = \frac{\bar{\beta}(1-\sigma)A}{\alpha} \kappa_t^{\sigma}.$$

It follows that the real steady state is unique and characterized by

$$\begin{aligned} \bar{\kappa}_R &= \left(\frac{\bar{\beta}(1-\sigma)A}{\alpha} \right)^{1/(1-\sigma)} \\ \bar{r}_R &= \frac{\sigma}{1-\sigma} \frac{\alpha}{\bar{\beta}}. \end{aligned}$$

There is over-accumulation of capital, or real steady-state allocations are dynamically inefficient, provided the growth rate of the economy is greater than the return on capital for L-consumers $\alpha > \beta_L \bar{r}_R$ or equivalently

$$\frac{\sigma}{1-\sigma} \frac{\beta_L}{\bar{\beta}} < 1. \tag{7}$$

To ease the exposition, we restrict attention to monetary equilibria in which L-consumers prefer to save by holding money and H-consumers by producing capital, i.e., $r_{t+1}\beta_L < p_{t+1} < r_{t+1}\beta_H$ for every t. Equation (2) becomes

$$\begin{aligned} \kappa_{t+1} &= \frac{\delta_H \beta_H (1+\gamma_{Ht}^y)}{\alpha} (1-\sigma) A \kappa_t^{\sigma} \\ \frac{M_t}{h_t P_t} &= \delta_L (1-\sigma)(1+\gamma_{Lt}^y) A \kappa_t^{\sigma} \\ \frac{M_t}{h_t P_t} &= \frac{M_{t-1}}{h_{t-1} P_{t-1}} \frac{p_t}{\alpha} + A(1-\sigma) \sum_i \delta_i \left(\frac{p_t}{\alpha} \kappa_{t-1}^{\sigma} \gamma_{it}^o + \kappa_t^{\sigma} \gamma_{it}^y \right). \end{aligned}$$

As in the general setup, printing money and distributing them to young H-consumers has two effects: First, young H-consumers have additional income and increase their savings for given prices. Therefore the capital-labor ratio increases because H-consumers invest more in capital production for given prices provided consumption when old is a normal good. Second, the return on money decreases.

Suppose that monetary policy is characterized by constant values for $\gamma = (\gamma_H^y, \gamma_L^y, \gamma_H^o, \gamma_L^o)$ for every t. Then monetary steady states are unique too, provided they exist, and are characterized by

$$\bar{\kappa}_M = \left(\frac{\delta_H\beta_H(1-\sigma)(1+\gamma_H^y)A}{\alpha}\right)^{1/(1-\sigma)}$$

$$\bar{p}_M = \alpha\frac{\delta_L-\delta_H\gamma_H^y}{\delta_L(1+\gamma_L^y+\gamma_L^o)+\delta_H\gamma_H^o}$$

$$\bar{r}_M = \frac{\sigma}{1-\sigma}\frac{\alpha}{\delta_H\beta_H(1+\gamma_H^y)}.$$

Only the money transferred to or from the young H-consumers matters for the steady-state capital-labor ratio. Monetary expansion increases the steady-state capital-labor ratio and output if and only if the new money is transferred to young H-consumers. The return on money and the inflation depend on the growth rate of the economy α and all drivers of the stock of money γ. In particular, $\alpha \geq \bar{p}$ if and only if $\delta_L(\gamma_L^y+\gamma_L^o)+\delta_H(\gamma_H^y+\gamma_H^o) \geq 0$.

The equilibrium allocations associated with monetary steady states are dynamically inefficient provided the return on capital is smaller than the growth rate $\alpha > \beta_H\bar{r}_M$ or equivalently

$$\frac{\sigma}{1-\sigma}\frac{1}{\delta_H(1+\gamma_H^y)} < 1. \tag{8}$$

Suppose

$$\delta_H+\gamma_H^y \leq \frac{\sigma}{1-\sigma} < \frac{\bar{\beta}}{\beta_L}.$$

Then equilibrium allocations associated with real steady states are dynamically inefficient and equilibrium allocations associated with monetary steady states are dynamically efficient according to Inequalities (7) and (8). Money restores efficiency because L-consumers switch their savings from investing in capital production to holding money.

Implications for monetary policy

We now look at the conditions for which output at monetary steady states with money printing is larger than output at real steady states. We restrict attention to monetary policies for which the associated allocation is dynamically efficient. Otherwise, the incremental output due to monetary policy would be welfare-reducing.

Theorems 1 and 2 state that there are real equilibria and real steady states. The real steady state is unique and, since $s(p) = 1$ for all p, the dynamics are well defined. Monetary steady states are unique provided they exist. Output at monetary steady states is larger than output in real steady states provided $\kappa_M/\kappa_R \geq 1$ or equivalently $\gamma_H^y \geq (\delta_L\beta_L)/(\delta_H\beta_H)$. However the condition does not ensure the existence of monetary steady states. Let the function $F : \mathbb{R}_{++} \to \mathbb{R}$ and the real number $\xi \in \mathbb{R}$ be defined by

$$F(\gamma) = \left(\frac{\delta_L}{\delta_H} - \gamma\right)(1+\gamma),$$

$$\xi = \left(\delta_H \beta_H \frac{1-\sigma}{\sigma}\right)^{-1} \left(\frac{\delta_L}{\delta_H}(1+\gamma_L^y+\gamma_L^o)+\gamma_H^o\right).$$

There are monetary steady states provided $\beta_L r_M < p_M < \beta_H r_M$ or equivalently

$$0 < F(\gamma_H^y)\xi^{-1} - \beta_L < \beta_H - \beta_L.$$

Let $G : \mathbb{R} \to \mathbb{R}$ be defined by $G(\gamma) = F(\gamma)\xi^{-1} - \beta$, and $(\gamma_1(\beta), \gamma_2(\beta))$ the two solutions to $G(\gamma) = 0$.

There are two cases, namely $\xi > 0$ and $\xi < 0$. Suppose $\xi > 0$. If $(2\delta_H)^{-2}\xi^{-1} < \beta_L$, then there is no γ_H^y for which a monetary steady state exists. If $\gamma_2(\beta_L) > (\delta_L\beta_L)/(\delta_H\beta_H)$, then there are monetary steady states and output is higher than at the real steady state with output being maximal for $\gamma_H^y = \gamma_2(\beta_L)$. Otherwise there are monetary steady states, but output is lower than at real steady states. Suppose $\xi < 0$. Then $\gamma_2(\beta_H) > (\delta_L\beta_L)/(\delta_H\beta_H)$ if and only if there are monetary steady states with $\gamma_H^y = \gamma_2(\beta_H)$ at which output is higher than at real steady states. Otherwise there are monetary steady states, but output is lower than at real steady state.

Whether monetary policies increase output or not depends on γ_H^y as well as $(\gamma_H^o, \gamma_L^y, \gamma_L^o)$ because the last three variables determine ξ and consequently the solution to $G(\gamma) = 0$. Output converges to its supremum for ξ converging to zero. One set of policies that result in ξ tending to zero is to tax away income of young L-consumers (γ_L^y close to minus one) and to balance transfers to old consumers ($\delta_L\gamma_L^o + \delta_H\gamma_H^o$ equal to zero). Obviously balanced transfers in form of zero transfers will bring inflation down to the inflation associated with printing money and distributing them to young H-consumers. For γ_H^y converging to δ_L/δ_H output at the monetary steady state is higher than output at the real steady state. The steady-state interest rate and the return on money converge to

$$\bar{r}_M = \alpha \frac{\sigma}{1-\sigma} \frac{1}{\beta_H}$$

$$\bar{p}_M = \alpha \frac{\sigma}{1-\sigma} \frac{\beta_L}{\beta_H}.$$

Therefore the supremum output is dynamically efficient provided $\sigma/(1-\sigma) \geq 1$ so the rate of return on capital production $\beta_H \bar{r}_M$ is equal to or higher than the growth rate α.

Finally, assume that transfers to the two kinds of consumers have to be identical, perhaps because productivity in capital production is private information or because of political reluctance to discriminate between the two kinds of consumers. Hence $\gamma_i^y = \gamma^y$ for some γ^y and $\gamma_i^o = 0$ for both i. Output is maximized for

$$\gamma^y = \frac{1}{\delta_H}\left(\delta_L - \frac{\sigma}{1-\sigma}\mu\right).$$

The steady-state interest rate is

$$\bar{r}_M = \alpha\frac{\sigma}{1-\sigma}\frac{\delta_H}{\delta_H\beta_H + (\sigma/(1-\sigma))\delta_L\beta_L}.$$

Hence the supremum output is dynamically efficient provided

$$\frac{\sigma}{1-\sigma}(1-\mu) \geq 1.$$

Welfare

We conclude the example by studying the welfare effect of monetary policies. In particular we show that there is a wide range of parameters for which monetary policies with relatively small money transfers improve welfare for both kinds of consumers. The reference scenario is the inactive monetary policy denoted γ^I and we consider two scenarios, both denoted γ^A: money transfers solely to young H-consumers; and, money transfers to both kinds of young consumers. In both scenarios there are no transfers to old consumers.

The ratio between steady state consumption for H-consumers between the two scenarios and the reference scenario is

$$\frac{c_H(\gamma^A)}{c_H(\gamma^I)} = (1+\gamma_H^y)^{\sigma/(1-\sigma)}.$$

Since the ratio is larger than one the welfare of H-consumers is increased in the two scenarios compared with the reference scenario. The ratio between steady state consumption for L-consumers between the two scenarios and the reference scenario is

$$\frac{c_L(\gamma^A)}{c_L(\gamma^I)} = \left(1-\frac{\delta_H}{\delta_L}\gamma_H^y\right)(1+\gamma_H^y)^{\sigma/(1-\sigma)}.$$

Therefore the ratio is larger than one if and only if

$$\frac{\delta_H}{\delta_L} < \frac{(1+\gamma_H^y)^{(1-\sigma)/\sigma} - 1}{(1+\gamma_H^y)^{(1-\sigma)/\sigma}\gamma_H^y}$$

for both scenarios. The expression on the right is decreasing in γ_H^y and by use of L'Hopital's rule it can be seen that it converges to $(1-\sigma)/\sigma$ as γ_H^y converges to zero. Hence if $\delta_L > \sigma$, then monetary policies in the two scenarios are welfare improving for both kinds of consumers provided γ_H^y is small. In Figure 3 the supremum value of γ_H^y, for which welfare of both kinds of consumers increases, is shown as a function of $(\delta_H/\delta_L, \sigma)$. For all parameters in the shown ranges there are welfare improving monetary policies. The range is decreasing in σ and increasing in δ_H/δ_L.

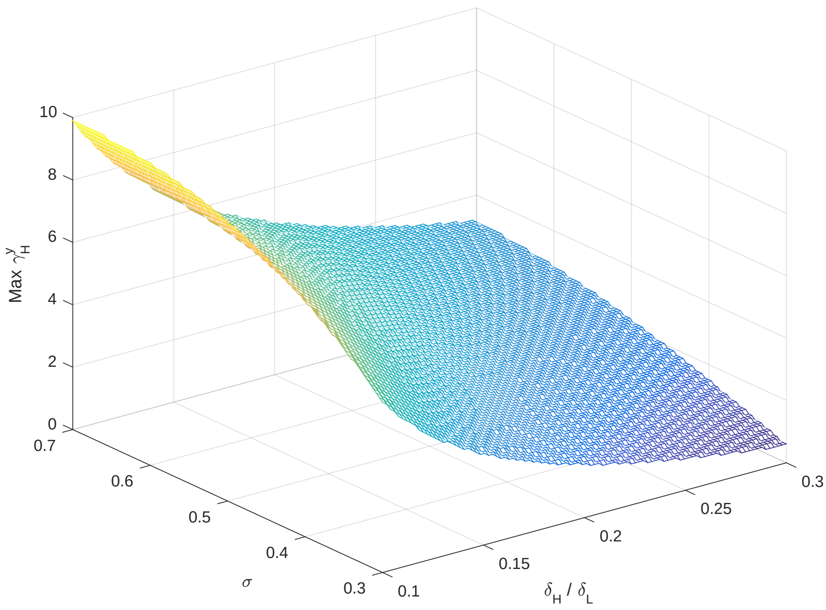


Figure 3: Monetary policies for which $c_L(\gamma^A)/c_L(\gamma^I) = 1$

7 Final remarks

Money matters in economies where efficient allocation of savings across investment opportunities is prevented by frictions in credit markets. The presence of money in itself does not help mitigate the effects of frictions. However we have shown that simple monetary policies in form of printing money and distributing them appropriately do help. Monetary policies in form of money taxes and money transfers can eliminate the effects of frictions, but these policies are institutionally more demanding.

We have demonstrated that different real outcomes can be obtained under the same 'inflation target'. Indeed the real effect of printing money depends on how they are distributed. Therefore the discussion of inflation targeting versus monetary targeting needs to be qualified: central bankers need to go beyond solely monitoring inflation.

We have considered a rather rudimentary asset market with two assets, namely investment in capital production and money. An additional financial asset would be needed to study quantitative easing, an unconventional policy recently adopted by some central banks consisting of exchanging money for financial assets. A natural candidate would be private credit, but the introduction of private credit would demand easing the frictions in the credit markets so part of investments could be used as collateral for loans.

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