Aggregation of opinions in networks of individuals and collectives*

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Abstract

We study the formation of opinions in a bipartite network with firm boards and directors. A director and a board are connected provided the director is a board member. Opinions are sets of beliefs about the likelihood of different states of the world tomorrow so an opinion is unambiguous in case it consists of a single belief and ambiguous otherwise. Our basic assumption is that boards as well as directors aggregate opinions of each other according to the Pareto principle: a production plan is better than another for a board (director) provided every director (board of which she is a member) finds it better. Opinions are stable provided interaction does not result in revision of opinions. We show that for connected networks: opinions are stable if and only if they are unambiguous and identical; and, repeated interaction leads to stable opinions. Hence, there will eventually be a society-wide, inter-subjective truth.

Keywords Boards · Directors · Networks · Pareto principle

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1 Introduction

We have looked at all 800 decisions taken in 2014 in general assembly meetings of the 40 largest French corporations (CAC 40) leaving out six decisions that were put on vote without the approval of the boards and obtained very little support (2-16 pct.). On average every decision was supported by 94.9% of the voters. The *consensus index* is 87.6 pct. meaning 87.6 pct. of the decisions were supported by at least 87.6 pct. of the voters. Decisions on collective issues had an index of 89.4 pct., decisions on individual compensation an index of 78.8 pct. and decisions on appointments an index of 87.5 pct. For anybody familiar with the social choice literature, these numbers are striking. Indeed an illustrative example of disagrement is m individuals sharing a cake where everybody just cares about the size of her own slice: for every allocation of the cake any $m-1$ individuals would agree to split the slice of last individual; and, the stability index would be zero.

The boards of corporations in CAC 40 have a total of 501 directors of which 57 are members of two boards, 14 are members of three boards and nobody is a member of four or more boards. In Figure 1 we show a network where corporations are nodes and common directors are links so there are 40 nodes and 99 links. A closer look at the boards in the

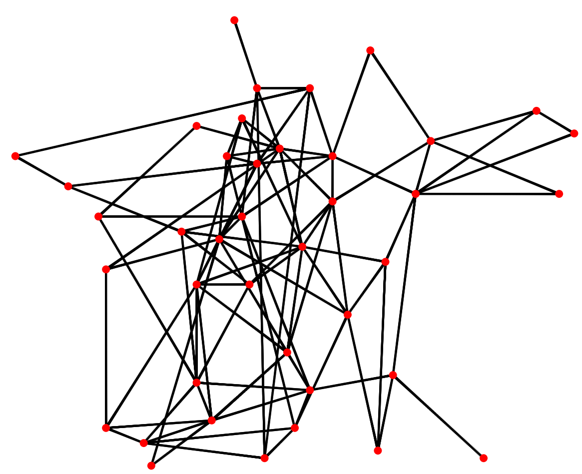

Figure 1: ***CAC 40***

CAC 40 corporations reveals that they are all connected directly or indirectly. Since the average number of links per firm is just below 2.5, connectedness of the network is striking. We develop a theory of formation of opinions in bipartite networks. Our theory uses one of the striking features, namely the connectedness of the network, to explain the other striking feature, namely the very high consensus index. Perhaps it would have been more natural to focus on either general assemblies or boards, but general assemblies appoint boards and boards put up proposals to general assemblies.

Several studies of networks of boards and directors have found that the networks tend to be connected (Burt, 2006; Davis, 1996). The median firm in Fortune 500 in the 1980s shared board members with seven other firms and some firms shared board members with 40 or more firms (Davis, 1996). It has been pointed out that firms benefit from being connected because thereby expertise and opinions of board members in other firms becomes indirectly available (Davis et al., 2002; Mace, 1971).

We study a world where decisions in firms have to be made today about production tomorrow. The state tomorrow is unknown so production is uncertain. In every firm a decision about production has to be taken. These decisions are equivalent to choosing beliefs for which firms should maximize profit. There are two types of actors, namely boards and directors. The decision in a firm is taken by the board that consists of some directors who can be members of one board or several boards. Every director has an opinion about how likely the different states are. These opinions can be unambiguous, in which case the director has a unique subjective belief about the likelihood of the different states, or ambiguous, in which case the director has multiple subjective beliefs.

Despite differences in opinions among directors, boards have to take decisions. We impose a weak condition, namely the Pareto principle, on aggregation processes of opinions in a board: If every director considers one production plan at least as good as another, and a least one director considers it better, then the board finds it better too. The opinions of boards are inter-subjective in that they are formed by a group of individuals, namely the directors, rather than by a single individual. Therefore the decisions taken in boards have a tinge of objectivity. We assume directors are affected or even overwhelmed by the decisions taken in boards: opinions of directors are outcomes of aggregation processes of the decisions in which they are involved. Like in boards, we impose the Pareto principle on aggregation processes of directors: If every board, in which a director is a member, considers one production plan at least as good as another, and at least one board considers it better, then the director finds it better too.

Since boards aggregate opinions of directors and directors aggregate opinions of boards, mutually interdependent aggregation processes take place in our world. Networks can be used to represent connections between firms and directors: a firm and a director are directly connected when the director is in the board of the considered firm. A firm and a director are indirectly connected when the director is in the board of a firm in which another director is a member of the board of another firm in which... another director is in the board of the considered firm. We focus on connected networks in which every firm and every director are directly or indirectly connected. However at the end of the paper we briefly discuss how our findings extend to disconnected networks.

Opinions are *stable* provided aggregation does not change them. Our main results concern stable opinions. We find that opinions are stable with respect to aggregation if and only if they are identical and unambiguous (Theorem 1). Hence stability of opinions is equivalent to a unique society-wide inter-subjective truth. Intuitively the Pareto principle implies aggregated opinions are convex combinations of opinions and every opinion is part of the aggregated opinion. Therefore aggregated opinions are inside the convex hull of opinions. Since every actor has an aggregated opinion of other aggregated opinions and so on, stable opinions are identical and unambiguous.

Since directors meet other directors in boards it could be argued that they should be affected by the opinions of other directors rather than decisions in boards. As a corollary to the characterization of stable opinions and beliefs we extend the characterization of stable opinions to include methodological individualism where directors are affected by other directors rather than boards as well as methodological collectivism where boards are affected by other boards rather than directors (Corollary 1).

We have modelled our world as an Actor-Network (Latour, 2007). It is flat: boards are formed by directors and directors are formed by boards; and, there is no aggregate level above them. Corollary 1 strengthen such an interpretation of our world. Opinions of actors depend on the opinions of their connections whose opinions depend on the opinions of their connections and so on: Directors and boards are their connections. Consequently it does not make sense to study an actor in isolation. Actors are treated symmetrically. They are collectives of collectives (Breiger, 1974; Simmel, 1955). Boards and directors aggregate opinions they experience from the other type of actor. However, rather than applying Actor-Network Theory to a specific topic like decisions in boards of the CAC 40 corporations in 2014 we have abstracted Actor-Network Theory by constructing a formal Actor-Network aimed at capturing the mutually interdependent aggregation processes in boards and directors. And rather than going into the details of how directors and boards interact and react, we merely assume that aggregation processes satisfy the Pareto principle.

The characterization of stable opinions is non-constructive like existence theorems in economics (Debreu, 1959), it is merely saying that there is no revision of opinions if and only if there is a unique society-wide inter-subjective truth. There is no explanation of how opinions become stable. Therefore we consider dynamic mutually interdependent aggregation processes where boards revise their opinions as a consequence of directors revising their opinions and directors revise their opinions as a consequence of boards revising their opinions and so on. Initial opinions are *consistent* provided initial opinions of boards (directors) are aggregated initial opinions of directors (boards). We find that dynamic mutually interdependent aggregation processes converge to stable opinions provided initial opinions are consistent (Theorem 2). As a corollary we show that if opinions are stable, then there are dynamic mutually interdependent aggregation processes that converge to these opinions and beliefs (Corollary 2). Hence stable opinions and limits of dynamic mutually interdependent aggregation processes are identical provided initial opinions are consistent.

Naturally aggregation can be interpreted as deliberative processes in boards and directors. Therefore stable opinions can be seen as reflective equilibria (Rawls 1971; Rorty, 1990; Daniels, 2011). In the context of reflective equilibrium, our two theorems are extremely strong. Deliberation leads to a unique society-wide inter-subjective truth so deliberation removes all ambiguity and resolves all conflicts. However in our world opinions are completely flexible in that actors are willing and able to revise them based on their experiences. And revision of opinions is in accordance with the Pareto principle so revision pushes toward agglomeration rather than polarization. As mentioned previously we discuss aggregation in disconnected networks at the end of the paper.

In Section 2 we outline the structure with time and uncertainty and describe the actors, boards and directors, and their relations. In Section 3 we introduce the Pareto principle for aggregation. In Section 4 we introduce our notion of stability of opinions, characterize stable opinions and outcomes of repeated social interaction. In Sections 6 and 7 we discuss our results and some extensions. Proofs are gathered in the Appendix.

2 Setup

The structure is outlined. There are two dates with no uncertainty at the first date and uncertainty at the second date. There are two types of actors, firms and directors. At the first date directors interact in boards of firms and boards interact in directors. At the second date production takes place.

Production and beliefs

Consider an economy with two dates $t \in \{0,1\}$ and uncertainty at the second date. There are S states at the second date $s \in \mathscr{S} = \{1,\ldots,S\}$. There is one commodity in every state.

There are J firms in the economy with $\mathscr{J} = \{1,\ldots,J\}$. Every firm j is characterized by its production set $Y_j \subset \mathbb{R}^S$ that is assumed to be non-empty, closed and convex. A production plan $y_j \in Y_j$ is efficient provided $Y_j \cap (\{y_j\} + \mathbb{R}^S_+) = \{y_j\}$. For every efficient production plans there is a belief $\nabla \in \mathbb{R}^S_+$ with $\nabla \neq 0$ such that $\nabla \cdot y_j \geq \nabla \cdot y'_j$ for all production plans $y'_j \in Y_j$. Therefore the choice of a production plan is equivalent to the choice of a belief. Let the unit simplex $\triangle^{S-1}_+$ in $\mathbb{R}^S$, where

$$\triangle^{S-1}_+ = \left\{ \nabla \in \mathbb{R}^S_+ \mid \sum_{s \in \mathscr{S}} \nabla^s = 1 \right\},$$

be the set of beliefs.

Directors and boards

There are I potential directors in the economy with $\mathscr{I} = \{1, \ldots, I\}$. Every firm j is governed by a board of directors $\mathscr{I}_j \subset \mathscr{I}$. Let $\mathscr{J}_i$ be the set of firms that have director i in their boards so $j \in \mathscr{J}_i$ if and only if $i \in \mathscr{I}_j$. Directors and boards have opinions in form of sets of beliefs that are used to assess production plans. Opinions are non-empty, closed and convex subsets of beliefs with $F_i \subset \triangle_+^{S-1}$ being the opinion of director i and $G_j \subset \triangle_+^{S-1}$ the opinion of firm j.

The opinion of director i (respectively board j) is unambiguous provided F_i (respectively G_j) contains a single belief, otherwise it is ambiguous. Actors with unambiguous opinions ∇ have complete preferences: y is at least as good as y' provided $\nabla \cdot y \geq \nabla \cdot y'$. However actors with ambiguous opinions H do not necessarily have complete preferences because for $\nabla, \nabla' \in H$ with $\nabla \neq \nabla'$ there are y and y' such that $\nabla \cdot y > \nabla \cdot y'$ and $\nabla' \cdot y < \nabla' \cdot y'$. Ambiguous decision makers à la Bewley (2002), for which y is at least as good as y' provided $\nabla \cdot y \geq \nabla \cdot y'$ for all $\nabla \in H$, have incomplete preferences. Ambiguous decision makers à la Gilboa and Schmeidler (1989), for which y is at least as good as y' provided $\min_{\nabla \in H} \nabla \cdot y \geq \min_{\nabla' \in H} \nabla' \cdot y'$, have complete preferences.

Consider a finite set of vectors $(v_k)_{k \in \mathscr{K}}$ with $v_k \in \mathbb{R}^S$. The *convex hull* of $(v_k)_{k \in \mathscr{K}}$, denoted $\operatorname{co}(v_k)_{k \in \mathscr{K}}$, is the set

$$\left\{ v \in \mathbb{R}^S \mid v = \sum_{k \in \mathscr{K}} \delta_k v_k \text{ where } \delta_k \geq 0 \text{ for every } k \text{ and } \sum_{k \in \mathscr{K}} \delta_k = 1 \right\}.$$

The *relative interior* of $\operatorname{co}(v_k)_{k \in \mathscr{K}}$, denoted $\operatorname{ri}\operatorname{co}(v_k)_{k \in \mathscr{K}}$, is the set

$$\left\{ v \in \mathbb{R}^S \mid v = \sum_{k \in \mathscr{K}} \delta_k v_k \text{ where } \delta_k > 0 \text{ for every } k \text{ and } \sum_{k \in \mathscr{K}} \delta_k = 1 \right\}.$$

For simplicity it is assumed that there is a finite set of beliefs $(\nabla_{im})_{m \in \mathscr{M}_i}$, respectively $(\nabla_{in})_{n \in \mathscr{N}_j}$, such that $F_i = \operatorname{co}(\nabla_{im})_{m \in \mathscr{M}_i}$ for every i, respectively $G_j = \operatorname{co}(\nabla_{jn})_{n \in \mathscr{N}_j}$ for every j.

3 The Pareto principle

As discussed in the introduction both types of actors aggregate opinions of each other. The Pareto principles for aggregation in boards and directors are introduced and discussed. It is a mild but fundamental condition on aggregation procedures. Indeed in Arrow (1951, 1963) the weak version of the Pareto principle is one of the conditions used to obtain the impossibility result in the General Possibility Theorem. Moreover it is a basic conditions in literature on joint aggregation of beliefs and tastes (Crès et al., 2012; Harsanyi, 1955; Hylland and Zeckhauser, 1979). Finally it is a founding condition in the literature on aggregation of judgements and logical aggregation theory (Kornhauser and Sager, 1986; List and Pettit, 2002; Mongin, 2012).

The Pareto principle across directors

Aggregation in boards of opinions of directors is assumed to satisfy the Pareto principle: If every director considers one production plan at least as good as another, and at least one director considers it better, then the board finds it better too.

Definition 1 *For sets of beliefs* $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})$*:*

- G_j *respects the* ***Pareto principle across directors (PPD)*** *provided that for every change* $\Delta y \in \mathbb{R}^S$*,* $\nabla_i{\cdot}\Delta y_j \geq 0$ *for all* $(i,\nabla_i) \in \mathscr{I}_j{\times}F_i$ *with* $>$ *for some* $(i,\nabla_i) \in \mathscr{I}_j{\times}F_i$ *implies* $\nabla_j{\cdot}\Delta y > 0$ *for every* $\nabla_j \in G_j$*.*
- $(G_j)_{j\in\mathscr{J}}$ *is* ***Pareto stable across directors (PSD)*** *provided that for every* j*,* G_j *respects PPD.*

The Pareto principle reflects that some director i would like to have some change of production, but not whether the rest of directors are indifferent or would like to have the change of production. Hence for ambiguous opinions the Pareto principle reflects some degree of optimism or ignorance.

Stability and value maximization

The Pareto principle across directors is equivalent to beliefs of boards being convex combinations of beliefs of directors with the belief of every directors being in the relative interior and the weight being positive on the belief of every director. Therefore no director is disregarded.

Lemma 1 *For* $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})$*, the set* G_j *respects PPD if and only if for every* $\nabla_j \in G_j$ *there is* $(\mu_{ij},\nabla_i)_{i\in\mathscr{I}_j}$ *with* $\mu_{ij} > 0$ *and* $\nabla_i \in \operatorname{ri} F_i$ *for every* i *such that* $\nabla_j = \sum_{i\in\mathscr{I}_j}\mu_{ij}\nabla_i$*.*

Since the weight on the belief of every director is positive, the Pareto principle implies minorities, even down to individual directors, have a say on the decisions of boards. Even though the Pareto principle does not imply that individual directors have veto power it is compatible with veto power.

The Pareto principle across boards

Since the opinions of boards are inter-subjective, directors are affected by opinions of boards as well as their own opinions. Therefore directors, just like boards, aggregate opinions. And like in boards, aggregation in directors of opinions of boards is assumed to satisfy the Pareto principle: If every board considers one production plan at least as good as another, and at least one board considers it better, then the director finds it better too.

Definition 2 *For sets of beliefs $((F_i)_{i\in\mathscr{I}}, (G_j)_{j\in\mathscr{J}})$:*

- *F_i respects the **Pareto principle across boards (PPB)** provided that for every change $\Delta y \in \mathbb{R}^S$, $\nabla_j\cdot\Delta y \geq 0$ for all $(j,\nabla_j) \in \mathscr{J}_i \times G_j \cup \{i\}\times F_i$ with $>$ for some $(j,\nabla_j) \in \mathscr{J}_i \times G_j \cup \{i\}\times F_i$ implies $\nabla_i\cdot\Delta y > 0$ for all $\nabla_i \in F_i$.*

- *$(F_i)_{i\in\mathscr{I}}$ is **Pareto stable across boards (PSB)** provided that for every i, F_i respects PPB.*

Stability and beliefs

The Pareto principle across boards is equivalent to beliefs of directors being convex combinations of beliefs of boards with the belief of every board being in the relative interior and the weight being positive on the belief of every board. And no board is disregarded.

Lemma 2 *For $((F_i)_{i\in\mathscr{I}}, (G_j)_{j\in\mathscr{J}})$, the set F_i respects PPB if and only if for every $\nabla_i \in F_i$ there is $(\nu_{ij}, \nabla_j)_{j\in\mathscr{J}_i}$ with $\nu_{ij} > 0$ and $\nabla_j \in ri\, G_j$ for every j such that $\nabla_i = \sum_{j\in\mathscr{J}_i} \nu_{ij}\nabla_j$.*

The Pareto principle for aggregation in directors can be seen as a formalization of the ideas that directors seek to have positive opinions about the decisions they experience in boards and that, by revising their opinions, directors try to minimize the tension between their own opinions and the opinions they experience in boards. On the one hand suppose the opinion of a director were outside the convex hull of the opinions of boards, then there would be a change of production which the director likes but no board likes. Hence such an opinion is neither positive about decisions nor minimizing tension between opinions. On the other hand suppose the opinion of a director were inside the convex hull of opinions of boards, then for every change of production if the director likes the change, then some board likes the change too.

4 Stability of opinions

Opinions of directors and firms are stable provided they respect the Pareto principle.

Definition 3 *Opinions* $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})$ *are* ***stable*** *provided* $(F_i)_{i\in\mathscr{I}}$ *is PSB and* $(G_j)_{j\in\mathscr{J}}$ *is PSD.*

Opinions are stable if and only if interaction and reaction do not lead to revision of opinions in neither boards nor directors. Social interaction makes no difference. The opinions of everybody is confirmed by the opinions of everybody else.

Consider a network obtained by the bipartite graph $\mathscr{A}$ with vertices $\mathscr{I}\cup\mathscr{J}$ and edges between i and j if and only if $i\in\mathscr{I}_j$ or equivalently $j\in\mathscr{J}_i$ and no other edges. Our world is flat: boards are formed by directors and directors are formed by boards; and, there is no aggregate level above them. Both directors and boards are their connections. Consequently it does not make sense to study an actor in isolation. Actors are treated symmetrical. Both directors and boards aggregate opinions they experience from the other type of actor. However compared with Actor-Network Theory (Latour, 2007) our network is abstract. We do not go into the specifics of how directors and boards interact and react, we merely assume that aggregation processes satisfy the Pareto principle.

For connected graphs, where every director and every board are connected directly or indirectly, opinions are stable if and only if they are identical and unambiguous.

Theorem 1 *Assume the graph* $\mathscr{A}$ *is connected. Then the sets of beliefs* $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})$ *are stable if and only if there is* $\nabla^*\in\triangle_+^{S-1}$ *such that*

$$((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) \;=\; \{\nabla^*\}^{I+J}.$$

For stable opinions all members of society have identical and unambiguous opinions. Social interaction will confirm the opinions of everybody in the most uninteresting way, namely by showing everybody that everybody else share her unambiguous opinion. In other words, there is a society-wide inter-subjective truth.

Theorem 1 offer theoretically support to empirical findings that connected boards creates an embeddedness for board decisions (Davis, 1996; Granovetter, 1985). Opionions in one firm or director becomes the starting point for opinions in other firms and directors and so on. Since our world is flat, there is no start and no end. However the structure of the network as well as how firms and directors aggregate opinions will determine how opinions evolve.

The characterization of stability in Theorem 1 is compatible with aggregation in either boards or directors provided either directors or boards aggregate opinions by taking the convex hull of the union of opinions of either boards or directors. The first scenario corresponds to methodological individualism and the second to the methodological collectivism.

Corollary 1 *Assume the graph* $\mathscr{A}$ *is connected. Suppose either:* $(F_i)_{i\in\mathscr{I}}$ *is PSB and* $G_j = co\cup_{i\in\mathscr{I}_j} F_i$ *for every* $j\in\mathscr{J}$*; or,* $F_i = co\cup_{j\in\mathscr{J}_i} G_j$ *for every* $i\in\mathscr{I}$ *and* $(G_j)_{j\in\mathscr{J}}$ *is PSD. Then there is* $\nabla^*\in\triangle_+^{S-1}$ *such that*

$$((F_i)_{i\in\mathscr{I}}, (G_j)_{j\in\mathscr{J}}) \;=\; \{\nabla^*\}^{I+J}.$$

The characterization in Corollary 1 is useful in that it allows methodological individualism where directors are affected by opinions of other directors rather than opinions of boards. The Pareto principle should be modified: If every director in every board of which director i is a member considers one production plan at least as good as another and at least one director considers it better, then director i finds it better too. Lemmas 1 and 3 and their proofs can be modified to show that the opinion F_i of director i respects the modified Pareto principle if and only if every belief of every director are convex combinations of all beliefs of all other directors she meets in boards. Formally F_i respects the modified Pareto principle if and only if

$$\text{For all } \nabla_i \in F_i, \nabla_i = \sum_{j\in\mathscr{J}_i}\sum_{k\in\mathscr{I}_j} \xi_{ik}\nabla_k \text{ with } \xi_{ik} > 0 \text{ and } \nabla_k \in \operatorname{ri} F_k \text{ for every } k.$$

Corollary 1 implies that if the opinion of director i respects PPB and G_j is the set of all convex combinations of all beliefs of every director, then opinions are stable if and only if there is a society-wide inter-subjective truth. Formally F_i respects PPB and $G_j = \operatorname{co} \cup_{k\in\mathscr{I}_j} F_k$ for every $j \in \mathscr{J}_i$, then $\nabla_i \in F_i$ if and only if

$$\nabla_i = \sum_{j\in\mathscr{J}_i}\sum_{k\in\mathscr{I}_j} \xi_{ik}\nabla_k \text{ with } \xi_{ik} > 0 \text{ and } \nabla_k \in \operatorname{ri} F_k \text{ for every } k.$$

Therefore Corollary 1 covers methodological individualism where the sole role of boards from the perspective of directors is as entities where directors meet other directors – as well as methodological collectivism where the sole role of directors from the perspective of boards is as entities where boards meet other boards.

5 Convergence of beliefs

Consider some structure on revisions of opinions, so there are functions mapping opinions into aggregated opinions. Then actors are described by their initial opinions and revision functions. Actors are not characterized by their opinions, but solely by how they revise their opinions. Therefore it can be argued that these functions are the closest we come to selves in our setup. Everything is revised as a result of social interaction, but there is structure in the aggregation principle underlying the revision. A position at one end of the spectrum is the position often taken in economics according to which preferences are fixed and therefore not subject to any kind of revision (Becker, 1976, 1996). A position at another end of the spectrum is that everything is constructed and therefore the result of revision (Gergen, 2015). Both positions leave little room for exploring the formation of opinions: according to the former nothing changes and according to the latter everything changes in a completely unstructured manner. Our revision processes reflect a middle position according to which opinions can change as a result of social interactions and reactions, but changes have some structure.

To formalize the middle position implicit in our setup two ingredients are needed. The first ingredient is functions mapping opinions into revised opinions. For directors these functions represent how directors process other opinions and consequently how they form their selves. For boards these functions can represent everything from mechanical aggregation like majority voting to deliberation of opinions. The second ingredient is a topology on opinions where the topology allows us to have a notion of the distance between two different opinions.

Let $\phi_i((G_j)_{j\in\mathscr{J}_i})$ be the revised opinion of director i in case the opinions in the boards that have director i as a member are $(G_j)_{j\in\mathscr{J}_i}$ and $\psi_j((F_i)_{i\in\mathscr{I}_j})$ the revised opinions in firm j in case the opinions of the directors of firm j are $(F_i)_{i\in\mathscr{I}_j}$. For simplicity it is assumed that there is a number $k\in\mathbb{N}$ such that if G_j (respectively F_i) is the convex hull of at most k vectors for every $j\in\mathscr{J}_i$ (respectively $i\in\mathscr{I}_j$), then $\phi_i((G_j)_{j\in\mathscr{J}_i})$ (respectively $\psi_j((F_i)_{i\in\mathscr{I}_j})$, is the convex hull of at most k vectors too.

Let $\mathscr{K}$ be the set of closed subsets of $\triangle_+^{S-1}$ that are convex hulls of at most k vectors. The set $\mathscr{K}$ is endowed with the Hausdorff distance $\rho:\mathscr{K}\times\mathscr{K}\to\mathbb{R}_+$ defined by

$$\rho(F,G) \;=\; \min\left\{\varepsilon\geq 0 \mid \max_{v\in F}\min_{w\in G}\|v-w\|, \max_{w\in G}\min_{v\in F}\|w-v\|\leq\varepsilon\right\}.$$

Then $\mathscr{K}$ is compact. Intuitively the Hausdorff distance for two sets is equal to the maximum of how far away it is possible to be in one of the sets from the other set.

It is assumed that: $\phi_i:\mathscr{K}^{\mathscr{J}_i}\to\mathscr{K}$ is continuous with $G_j\subset G'_j$ for every $j\in\mathscr{J}_i$ implying $\phi_i((G_j)_{j\in\mathscr{J}_i})\subset\phi_i((G'_j)_{j\in\mathscr{J}_i})$ for every i; and, $\psi_j:\mathscr{K}^{\mathscr{I}_j}\to\mathscr{K}$ is continuous with $F_i\subset F'_i$ for every $i\in\mathscr{I}_j$ implying $\psi_j((F_i)_{i\in\mathscr{I}_j})\subset\psi_j((F'_i)_{i\in\mathscr{I}_j})$ for every j. Moreover it is assumed that ϕ_i respects PPB for every i and ψ_j respects PPD for every j: $\nabla_i\in\phi_i((G_j)_{j\in\mathscr{J}_i})$ implies there is $(\nu_{ij},\nabla_j)_{j\in\mathscr{J}_i}$ with $\nu_{ij}>0$ and $\nabla_j\in\operatorname{ri}G_j$ for every j such that $\nabla_i=\sum_{j\in\mathscr{J}_i}\nu_{ij}\nabla_i$; and, $\nabla_j\in\psi_j((F_i)_{i\in\mathscr{I}_j})$ implies there is $(\mu_{ij},\nabla_i)_{i\in\mathscr{I}_j}$ with $\mu_{ij}>0$ and $\nabla_i\in\operatorname{ri}F_i$ for every i such that $\nabla_j=\sum_{i\in\mathscr{I}_j}\mu_{ij}\nabla_i$.

Repeated revision of opinions leads to elimination of ambiguity independently of initial opinions. However since actors of one type base their revisions of opinions solely on opinions of actors of the other type, opinions can oscillate. The reason is that there is no interaction between opinions.

Lemma 3 *Assume the graph $\mathscr{A}$ is connected. Then for all sets of beliefs $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})\in\mathscr{K}^{I+J}$, there are $\nabla^*\in ri\,co\cup_{i\in\mathscr{I}}F_i$ and $\nabla^{**}\in ri\,co\cup_{j\in G_j}G_j$ such that*

$$\begin{aligned}\lim_{n\to\infty}((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})^{2n}((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) &= \{\nabla^*\}^I\times\{\nabla^{**}\}^J\\ \lim_{n\to\infty}((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})^{2n+1}((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) &= \{\nabla^{**}\}^I\times\{\nabla^*\}^J.\end{aligned}$$

Repeated social interaction leads to elimination of ambiguity: all members of society have unambiguous, possible oscillating, opinions. Initial opinions of directors and firms matter for convergence of opinions.

In case of one director and one board with unambiguous initial opinions F and G, the first round of revisions lead to opinions being switched $G=\phi(F)$ for the director and $G=\psi(F)$ for the firm, the second round of revisions lead to opinions being switched back $F=\phi(G)$ for the director and $G=\psi(F)$ for the firm and so on. Opinions simply oscillate and there is no convergence. However there will eventually be a society-wide inter-subjective truth provided initial opinions are consistent.

Suppose initial opinions are *consistent* in the sense that either initial opinions of directors are revised opinions of firms or initial opinions of firms are revised opinions of directors. Then repeated revision of opinions leads to opinions converging to stable opinions.

Theorem 2 *Assume the graph $\mathscr{A}$ is connected. Then for all sets of beliefs $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})\in\mathscr{K}^{I+J}$ with $F_i\subset\phi_j((G_j)_{j\in\mathscr{J}_i})$ for every i or $G_j\subset\psi_j((F_i)_{i\in\mathscr{I}_j})$ for every j there is $\nabla^*\in ri\,co\cup_{i\in\mathscr{I}}F_i$ or $\nabla^*\in ri\,co\cup_{j\in G_j}G_j$ such that*

$$\lim_{n\to\infty}((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})^{n}((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) = \{\nabla^*\}^{I+J}.$$

History dependence in revisions of opinions for some type of actors would ensure convergence just like consistent initial opinions. The Pareto principle should be modified for these actors: If one production plan is at least as good as another according to every opinion (including the actor's own opinion) an actor experiences and it is better according to at least one opinion, then the actor should find it better too. Applied to the case with one director and one firm, an actor with history dependent revision of opinions would form an opinion between her own and the other actor's initial opinions rather than switching to the initial opinion of the other actor.

Some possibilities of escaping society-wide inter-subjective truths in the presence of consistent initial opinions or history dependent revisions are: disconnected clusters of directors and firms; or, random shocks in revisions of opinions. Both possibilities are discussed in detail in Section 7.

Corollary 2 *Assume the graph $\mathscr{A}$ is connected. Then for all sets of beliefs $((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}})\in\mathscr{K}^{I+J}$ and $\nabla^*\in ri\,co\cup_{i\in\mathscr{I}}F_i$ and $\nabla^{**}\in ri\,co\cup_{j\in G_j}G_j$ there are $((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})$ such that*

$$\begin{aligned}\lim_{n\to\infty}((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})^{2n}((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) &= \{\nabla^*\}^I\times\{\nabla^{**}\}^J\\ \lim_{n\to\infty}((\phi_i)_{i\in\mathscr{I}},(\psi_j)_{j\in\mathscr{J}})^{2n+1}((F_i)_{i\in\mathscr{I}},(G_j)_{j\in\mathscr{J}}) &= \{\nabla^{**}\}^I\times\{\nabla^*\}^J.\end{aligned}$$

6 Discussion of our results

In a setup with directors and firms we have combined two ideas, namely that both boards and directors revise their opinions and that revisions are compatible with the Pareto principle. Thereby we have obtained a foundation for DeGroot learning (DeGroot, 1974) as shown in Lemmas 1 and 3 and Theorem 2. The outcome of the repeated revision of opinions is that they become identical and unambiguous or stable as shown in Theorems 1 and 2. Hence social interaction establishes a society-wide inter-subjective truth provided all members of society are connected and initial opinions are consistent or some revisions are history dependent.

The Pareto principle within boards is quite standard. It is simply a formalization of the idea that aggregation of opinions in boards should not lead to opinions at odds with the opinion of every director. The Pareto principle within directors is not standard, at least not in economics. However it is a formalization of the ideas that individuals seek to have positive opinions about what they experience and that individuals try to minimize tension between the opinions they experience and their own opinions.

For stable opinions all actors have the same unambiguous opinion so every firm maximizes its profit for the same belief and the vector is identical to the belief of every director. Market completeness does not matter for the result: markets can be complete (at least as many linearly independent production plans as states) or incomplete (fewer linearly independent production plans than states). In standard economics it matters a lot whether markets are complete or incomplete (Magill and Quinzii, 1996; Tvede and Crès, 2005). In the former case all actors end up with identical beliefs. In the latter case actors typically end up with different beliefs. Since actors do not revise opinions in standard economics, it does not matter whether actors are connected or not.

7 Extensions

In relation to Theorem 1, if some directors and some firms are not connected then the graph $\mathscr{A}$ is not connected. Indeed it consists of a number of connected components $\mathscr{A}=\cup_{b\in\mathscr{B}}\mathscr{A}_b$

where every director and every firm is in exactly one component. Our findings for connected graphs generalize immediately to every connected component. Therefore in every connected component opinions are stable if and only if they are identical and unambiguous. However opinions can very well differ across connected components. But they differ exactly because there is no social interaction across connected components.

In the present paper we have considered board membership exogenous. Hence the graph $\mathscr{A}$ is fixed. It is natural to make connections endogenous by letting consumers choose which firms to be connected to through their choice of portfolios and firms, possible through their assembly of shareholders, their boards. A new feature coming from endogenous network formation is that small changes in portfolios can have a big impact on the outcome of repeated revision of opinions. Imagine that given the choices of portfolios there are two connected components $\mathscr{A}_1$ and $\mathscr{A}_2$ and that one shareholder in the first component has zero units of shares in some firm in the second component. Now for a whatever small increment in the units of shares in the firm in the second component the graph becomes connected. Therefore the society would jump from having two different inter-subjective truths, namely one for every connected component, to having a unique society-wide inter-subjective truth.

In relation to Theorem 2, if there were random shocks in the revision of opinions, then the notion of stability would have to be changed to a probability distribution on opinions. We conjecture that the probability distribution on opinions would reflect the randomness in the revision. Small shocks will result in more or less identical small sets of beliefs measured by the Hausdorff distance being very likely, so opinions will be more or less identical and unambiguous most of the time. Large shocks will result in different large sets of beliefs being very likely so opinions will be very different and very ambiguous most of the time.

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Appendix: proofs

Proof of Lemma 1:

According to Theorem 22.2 in Rockafellar (1970) either, corresponding to G_j does not respect PPD, there is $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla_{im}\cdot\Delta y &\geq 0 \text{ for every } (i,m) \in \mathscr{I}_j \times \mathscr{M}_i \\ \textstyle\sum_{(i,m)\in\mathscr{I}_j\times\mathscr{M}_j} \nabla_{im}\cdot\Delta y &> 0 \\ \nabla_j\cdot\Delta y &\leq 0 \text{ for some } \nabla_j \in G_j \end{aligned}$$

or, corresponding to G_j does respect PPD, for all $\nabla_j \in G_j$ there are $(\alpha_{im})_{(i,m)\in\mathscr{I}_j\times\mathscr{M}_i}$ with $\alpha_{im} \geq 0$ for every (i,m), $\beta > 0$ and $\gamma \geq 0$ such that

$$\textstyle\sum_{(i,m)\in\mathscr{I}_j\times\mathscr{M}_i} \alpha_{im}\nabla_{im} + \beta \sum_{(i,m)\in\mathscr{I}_j\times\mathscr{M}_i} \nabla_{im} - \gamma\nabla_j \;=\; 0.$$

Clearly $\alpha_{im} \geq 0$ and $\nabla_{im} \in \triangle_+^{S-1}$ for every (i,m) and $\beta > 0$ implies $\gamma > 0$. Therefore

$$\begin{aligned} \nabla_j &= \sum_{(i,m)\in\mathscr{I}_j\times\mathscr{M}_i} \frac{\alpha_{im}+\beta}{\gamma}\nabla_{im} \\ &= \sum_{i\in\mathscr{I}_j} \frac{\sum_{p\in\mathscr{M}_i}(\alpha_{ip}+\beta)}{\gamma} \sum_{m\in\mathscr{M}_i} \frac{\alpha_{im}+\beta}{\sum_{p\in\mathscr{M}_i}(\alpha_{ip}+\beta)}\nabla_{im} \\ &= \sum_{i\in\mathscr{I}_j} \mu_{ij}\nabla_i \end{aligned}$$

where

$$\mu_{ij} \;=\; \sum_{p\in\mathscr{M}_i}\frac{\alpha_{ip}+\beta}{\gamma} \;\text{ and }\; \nabla_i \;=\; \sum_{m\in\mathscr{M}_i}\frac{\alpha_{im}+\beta}{\sum_{p\in\mathscr{M}_i}(\alpha_{ip}+\beta)}\nabla_{im}.$$

Clearly $\mu_{ij} > 0$ and $\nabla_i \in \operatorname{ri} F_i$ for every $i \in \mathscr{I}_j$.

Proof of Lemma 3:

According to Theorem 22.2 in Rockafellar (1970) either, corresponding to F_i does not respect PPB, there is $\Delta y \in \mathbb{R}^S$ such that

$$\begin{aligned} \nabla_{jn}\cdot\Delta y &\geq 0 \text{ for every } (j,n) \in \mathscr{J}_i\times\mathscr{N}_j \\ \textstyle\sum_{(j,n)\in\mathscr{J}_i\times\mathscr{N}_j}\nabla_{jn}\cdot\Delta y &> 0 \\ \nabla_i\cdot\Delta y &\leq 0 \text{ for some } \nabla_i \in F_i \end{aligned}$$

or, corresponding to F_i does respect PPB, for all $\nabla_i \in F_i$ there are $(\alpha_{jn})_{(j,n)\in\mathscr{J}_i\times\mathscr{N}_j}$ with $\alpha_{jn} \geq 0$ for every (j,n), $\beta > 0$ and $\gamma \geq 0$ such that

$$\textstyle\sum_{(j,n)\in\mathscr{J}_i\times\mathscr{N}_j}\alpha_{jn}\nabla_{jn} + \beta\sum_{(j,n)\in\mathscr{J}_i\times\mathscr{N}_j}\nabla_{jn} - \gamma\nabla_i \;=\; 0.$$

Clearly $\alpha_{jn} \geq 0$ and $\nabla_{jn} \in \triangle_+^{S-1}$ for every (j,n) and $\beta > 0$ implies $\gamma > 0$. Therefore

$$\begin{aligned} \nabla_i &= \sum_{(j,n)\in\mathscr{J}_i\times\mathscr{N}_j}\frac{\alpha_{jn}+\beta}{\gamma}\nabla_{jn} \\ &= \sum_{j\in\mathscr{J}_i}\frac{\sum_{p\in\mathscr{N}_j}(\alpha_{jp}+\beta)}{\gamma}\sum_{n\in\mathscr{N}_j}\frac{\alpha_{jn}+\beta}{\sum_{p\in\mathscr{N}_j}(\alpha_{jp}+\beta)}\nabla_{jn} \\ &= \sum_{j\in\mathscr{J}_i}\nu_{ij}\nabla_j \end{aligned}$$

where

$$\nu_{ij} \;=\; \sum_{p\in\mathscr{N}_j}\frac{\alpha_{jp}+\beta}{\gamma} \;\text{ and }\; \nabla_j \;=\; \sum_{n\in\mathscr{N}_j}\frac{\alpha_{jn}+\beta}{\sum_{p\in\mathscr{N}_j}(\alpha_{jp}+\beta)}\nabla_{jn}.$$

Clearly $\nu_{ij} > 0$ and $\nabla_j \in \operatorname{ri} G_j$ for every $j \in \mathscr{J}_i$.

Proof of Theorem 1:

Let $P_{\mathscr{I}}$ be the convex hull of $\cup_{i\in\mathscr{I}}F_i$ and $P_{\mathscr{J}}$ the convex hull of $\cup_{j\in\mathscr{J}}G_j$. Then $P_{\mathscr{I}} = P_{\mathscr{J}}$ according to Lemmas 1 and 3.

Suppose p is an extreme points of $P_{\mathscr{I}} = P_{\mathscr{J}}$ and let $\mathscr{I}(p) = \{\, i \in \mathscr{I} \mid p \in F_i \,\}$ and $\mathscr{J}(p) = \{\, j \in \mathscr{J} \mid p \in G_j \,\}$. Then $\mathscr{I}(p), \mathscr{J}(p) \neq \emptyset$ by construction. Moreover for every

$i \in \mathscr{I}(p)$, $\mathscr{J}_i \subset \mathscr{J}(p)$ with $G_k = \{p\}$ for every $k \in \mathscr{J}_i$ according to Lemma 3 and for every $j \in \mathscr{J}(p)$, $\mathscr{I}_j \subset \mathscr{I}(p)$ with $F_k = \{p\}$ for every $k \in \mathscr{I}_j$ according to Lemma 1 because p is an extreme point of $P_{\mathscr{I}} = P_{\mathscr{J}}$. Therefore $F_i = \{p\}$ for every $i \in \mathscr{I}(p)$ according to Lemma 3 and $G_j = \{p\}$ for every $j \in \mathscr{J}(p)$ according to Lemma 1. Since $\mathscr{A}$ is connected, $\mathscr{I}(p) = \mathscr{I}$ and $\mathscr{J}(p) = \mathscr{J}$. □

Proof of Lemma 3:

Let $\mathscr{L}$ be the set of non-empty and closed subsets L of $\triangle_+^{S-1}$ so $\mathscr{K} \subset \mathscr{L}$. The set $\mathscr{L}$ is endowed with the Hausdorff distance $\rho : \mathscr{L} \times \mathscr{L} \to \mathbb{R}_+$. Then $\mathscr{L}$ is compact. Let $\delta : \mathscr{L} \to \mathbb{R}_+$ be the diameter: $\rho(L) = \max_{v,w \in L} \|v - w\|$. For every $\varepsilon > 0$ the subset $\mathscr{L}_\varepsilon$ of sets in $\mathscr{L}$ with diameter at least ε defined by

$$\mathscr{L}_\varepsilon \;=\; \{L \in \mathscr{L} \mid \delta(L) \geq \varepsilon\}$$

is compact.

Let $\Gamma : \mathscr{K}^{\mathscr{I}} \to \mathscr{K}^{\mathscr{I}}$ be defined by $\Gamma_k((F_i)_i) = \phi_k((\psi_j((F_i)_{i \in \mathscr{I}_j}))_{j \in \mathscr{J}_i})$. Then $\delta(\cup_k \Gamma_k((F_i)_i)) \leq \delta(\cup_i F_i)$ because ϕ_i respects PPB for every i and ψ_j respects PPD for every j. Suppose p is an extreme point of $\cup_i F_i$. Let

$$\begin{aligned}
\mathscr{I}^0 &= \{i \in \mathscr{I} \mid F_i = \{p\}\} \\
\mathscr{J}^0 &= \{j \in \mathscr{J} \mid \mathscr{I}_j \subset \mathscr{I}^0\} \\
\mathscr{I}^1 &= \{i \in \mathscr{I}^0 \mid \mathscr{J}_i \subset \mathscr{J}^0\} \\
\mathscr{J}^1 &= \{j \in \mathscr{J}^0 \mid \mathscr{I}_j \subset \mathscr{I}^1\} \\
&\;\vdots \\
\mathscr{I}^{n+1} &= \{i \in \mathscr{I}^m \mid \mathscr{J}_i \subset \mathscr{J}^n\} \\
\mathscr{J}^{n+1} &= \{j \in \mathscr{J}^m \mid \mathscr{I}_j \subset \mathscr{I}^n\} \\
&\;\vdots
\end{aligned}$$

Then either $\mathscr{I}^n = \mathscr{I}$ and $\mathscr{J}^n = \mathscr{J}$ for every n or there is $N \in \mathbb{N}$ such that $\mathscr{I}^n = \mathscr{J}^n = \emptyset$ for every $n \geq N$. In the first case $\Gamma^{2n}((F_i) = \{p\}$. In the second case there is $N \in \mathbb{N}$ such that for every extreme point p of $\cup_i F_i$ and $n \geq N$, $p \notin \Gamma^n(F_i)$ so $\delta(\cup_k \Gamma_k^n((F_i)_i)) < \delta(\cup_i F_i)$. Since Γ is continuous and $\mathscr{L}_\varepsilon$ is compact, there $\alpha_\varepsilon \in]0,1[$ such that $\delta(\cup_k \Gamma_k^n((F_i)_i)) \leq \alpha_\varepsilon \delta(\cup_i F_i)$ for all $(F_i)_i$ with $F_i \in \mathscr{L}_\varepsilon$. Therefore $\lim_{n\to\infty} \delta(\cup_k \Gamma_k^n((F_i)_i)) = 0$ so there is $\nabla^* \in \operatorname{ri}\operatorname{co} \cup_{i \in \mathscr{I}} F_i$ such that $\lim_{n\to\infty} \Gamma^n((F_i)_i) = \{\nabla^*\}^I$. □