Walrasian Lemons Markets*

WORK IN PROGRESS

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Abstract

We study Walrasian trade in lemons markets. There is an equal mass of buyers and sellers in a market. Sellers can produce one unit of an indivisible good and buyers can consume one unit of this good. There are finitely many different types of good and sellers are privately informed about the type of good they can produce. We define competitive equilibria. Unlike most models of Walrasian trade with adverse selection, we allow trade to take place at different prices. We show that competitive equilibria exist and characterize them. We then show that competitive equilibria are generically constrained inefficient when adverse selection is severe enough to prevent the first-best from being achieved. We also show that one can implement the second best by means of budget balanced taxes and transfers to sellers. We relate our work to the work on competitive search with adverse selection.

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1 Introduction

The more common approach to study trade in markets with adverse selection is to model the market as a game where the details of the trading process are completely specified and characterize the equilibria of this game. A drawback of this approach is that the outcome of the trading process is in general sensitive to how one models the process of trade.

An alternative approach to study trade in markets with adverse selection is to abstract from the details of the trading process and consider a Walrasian model of trade. Most models of Walrasian trade with adverse selection assume that all trades take place at the same price, though. This rules out *a priori* the ability of prices to play a signalling role, which is an important limitation when considering trade with adverse selection.

In this paper we develop a model of Walrasian trade of an indivisible good in the presence of adverse selection in which trade can take place at different prices. We define competitive equilibria, show existence, and characterize them. We then show that when adverse selection is severe enough to prevent the first-best from being implemented, competitive equilibria are generically constrained inefficient. We also show that one can implement the second best by means of budget balanced taxes and transfers to sellers. We conclude by showing that competitive equilibrium allocations can be approximated by the equilibrium allocations of a dynamic matching and bargaining game when sellers make take-it-or-leave-it offers to buyers.

We start in Section 2 by introducing the environment and defining competitive equilibria. Our environment is related to the environment in Gale (1996) but our analysis differs from Gale's analysis in some important ways. We discuss these differences in detail in Section 2. We also discuss the relationship between our work the work of Guerrieri, Shimer, and Wright (2010) on competitive search with adverse selection. In Section 3 we establish a number of results about competitive equilibria, including equilibrium existence. In Section 4 we establish some further results about competitive equilibria. In Section 5 we discuss constrained efficiency of competitive equilibria. Section 6 concludes and the Appendix contains omitted proofs and details.

2 Environment and Equilibria

We first describe the physical environment and then define competitive equilibria. We conclude by discussing the relationship between our environment and the environments in Gale (1996) and Guerrieri, Shimer, and Wright (2010).

2.1 Physical Environment

There is an equal mass of buyers and sellers in the market. Each seller can produce one unit of an indivisible good and each buyer wants to consume one unit of the good. Sellers differ in the type θ of good they can produce, which is their private information. We refer to a seller who produces a good of type θ as a type-θ seller. Let $\Theta = \{\theta_1, \ldots, \theta_N\}$, with $N \geq 2$, be the set of possible types of good, ordered from lowest to highest, and $f_\theta \in (0, 1)$ be the fraction of type-θ sellers. The value of a good of type θ is v^s_θ for sellers and $v^b_\theta > v^s_\theta$ for buyers. So, gains from trade are positive for all types of good. Moreover, v^s_θ and v^b_θ are strictly increasing in θ.[1] Finally, preferences are quasi-linear. The payoff to a buyer who pays price $p \in \mathbb{R}_+$ for a good of type θ is $v^b_\theta - p$. The payoff to a type-θ seller who sells his good for price p is $p - v^s_\theta$.

Buyers and sellers in the market choose the price at which they would like to trade. To each price $p \in \mathbb{R}_+$ there is associated a sub-market, or trading post, in which buyers and sellers meet to transact at this price. Even though not necessary for our analysis, one can think that buyers and sellers who visit a given sub-market meet bilaterally to trade.[2] Buyers and sellers also have the option of not trading, which we interpret as the choice of going to a sub-market, n, where the probability of trade is zero. In what follows, we do not make a distinction between a price $p \in \mathbb{R}_+$ and the sub-market in which buyers and sellers meet to trade at this price and let $\mathbb{R}^* = \mathbb{R}_+ \cup \{n\}$ denote the set of all sub-markets.

Adverse selection is *severe* if

$$\sum_{\theta \in \Theta} f_\theta v^b_\theta < v^s_{\theta_N}.$$

In this case, not all trades can take place at the same price. Even though we do not impose severe

[1] Gains from trade need not be increasing in θ, though.

[2] This is useful later on when we discuss the relation between our work and Guerrieri, Shimer, and Wright (2010).

adverse selection *a priori*, it is clear that this is the more interesting case to analyze.

2.2 Competitive Equilibria

A competitive equilibrium specifies the trading decisions of buyers and sellers and the terms of trade in the market. The latter are determined by the outcomes of trading decisions and buyer beliefs. We describe trading decisions, trading outcomes, and (buyer) beliefs in what follows.

A *trading rule* describes the trading decisions of buyers and sellers, that is, the sub-markets to which they direct their trade. It is given by a list $\lambda = (\lambda^b, (\lambda^s_\theta)_{\theta\in\Theta})$ of probability measures on $\mathbb{R}^*$ such that $\lambda^b(P)$ and $\lambda^s_\theta(P)$ are, respectively, the probabilities that buyers and type-θ sellers go to a sub-market in the set $P \subseteq \mathbb{R}^*$ of sub-markets.[3] A *trading outcome* describes the outcome of trading decisions. It is given by a pair $q = (q^b, q^s)$ of functions from $\mathbb{R}^*$ into $[0, 1]$ such that $q^b(p)$ and $q^s(p)$ are, respectively, the probabilities that buyers and sellers trade in case they go to sub-market p, with $q^b(n) = q^s(n) = 0$. We can interpret $q^b(p)$ as the probability that a buyer who goes to sub-market p is matched with a seller to trade. Likewise, we can interpret $q^s(p)$ as the probability that a seller who goes to sub-market p is matched with a buyer to trade. Finally, a *belief* is a function $\beta : \mathbb{R}_+ \to \Delta_N$, where Δ_N is the unit simplex in $\mathbb{R}^N$, such that $\beta(p)$ is the buyers' belief about the type of good transacted in sub-market p. We let $\beta_\theta(p)$ be the probability that $\beta(p)$ assigns to the event that the good transacted in sub-market p is of type θ.

Given a trading rule λ and a trading outcome q, let μ^b be the measure on $\mathbb{R}^*$ such that

$$\mu^b(P) = \int_P q^b(p)d\lambda^b(p).$$

Similarly, for each $\theta \in \Theta$, let μ^s_θ be the measure on $\mathbb{R}^*$ such that

$$\mu^s_\theta(P) = \int_P q^s(p)d\lambda^s_\theta(p).$$

By construction, $\mu^b(P)$ is the probability a buyer trades in the set $P \subseteq \mathbb{R}_+$ of sub-markets, while $\mu^s_\theta(P)$ is the probability a type-θ seller trades in the same set of sub-markets. The supports of μ^b and μ^s_θ are contained in $\mathbb{R}_+$, as no trade takes place in sub-market n by assumption.

[3]The measures λ^b and $(\lambda^s_\theta)_{\theta\in\Theta}$ are defined on the Borel sets of $\mathbb{R}^*$—the topology on $\mathbb{R}^*$ is the one-point compactification of $\mathbb{R}_+$. In what follows, we take any measure to be a Borel measure and any function to be Borel measurable.

A competitive equilibrium also specifies a vector $U = (U^b, (U^s_\theta)_{\theta\in\Theta})$ of payoffs such that U^b is the payoff to buyers and U^s_θ is the payoff to type-θ sellers.

Definition. *A list (λ, q, β, U), where λ is a trading rule, q is a trading outcome, β is a belief, and U is a vector of payoffs is a competitive equilibrium if it satisfies the following conditions.*

1. Buyer optimality: *For each $p \in \mathbb{R}^*$,*

$$U^b \geq q^b(p)\left(\sum_\theta \beta_\theta(p) v^b_\theta - p\right), \text{ with equality holding } \lambda^b\text{-almost surely.} \tag{1}$$

2. Seller optimality: *For each $\theta \in \Theta$ and $p \in \mathbb{R}^*$,*

$$U^s_\theta \geq q^s(p)(p - v^s_\theta), \text{ with equality holding } \lambda^s_\theta\text{-almost surely.} \tag{2}$$

3. Market clearing: *For each $\theta \in \Theta$ and $P \subseteq \mathbb{R}_+$,*

$$\int_P \beta_\theta(p) d\mu^b(p) = f_\theta \mu^s_\theta(P). \tag{3}$$

4. Monotonic Trading: *For each $p, p' \in \mathbb{R}_+$, $q^s(p') \leq q^s(p)$ implies that $q^b(p') \geq q^b(p)$.*

Buyer and seller optimality are straightforward to interpret. They require that buyers and sellers behave optimally taking the terms of trade as given. The terms of trade for a buyer in sub-market $p \in \mathbb{R}_+$ are described by the trading probability $q^b(p)$ and the belief $\beta(p)$. The terms of trade for a seller in sub-market $p \in \mathbb{R}_+$ are described by the trading probability $q^s(p)$.

In order to understand market clearing, notice that for any set $P \subseteq \mathbb{R}_+$ of sub-markets, the right side of (3) is the mass of type-θ sellers who sell their good in P, while the left side of (3) is the mass of buyers who purchase a good of type θ in the same set of sub-markets. If we let $\overline{\mu}^s$ be the measure on $\mathbb{R}^*$ such that

$$\overline{\mu}^s = \sum_{\theta\in\Theta} f_\theta \mu^s_\theta,$$

then $\overline{\mu}^s(P)$ is the mass of sellers who trade in P. Market clearing implies that μ^b and $\overline{\mu}^s$ coincide. The common support $\mathcal{P}$ of these measures is the set of sub-markets that open in equilibrium.

It follows from market clearing that

$$\int_P \beta_\theta(p) d\overline{\mu}^s(p) = f_\theta \mu_\theta^s(P) \tag{4}$$

for all $P \subseteq \mathbb{R}_+$ and $\theta \in \Theta$. Therefore, β satisfies Bayes' rule in the sub-markets that open in equilibrium.[4] In particular, if the set $\mathcal{P}$ of sub-markets that open in equilibrium is finite, then (4) implies that

$$\beta_\theta(p) = \frac{f_\theta \mu_\theta^s(\{p\})}{\sum_{\theta' \in \Theta} f_{\theta'} \mu_{\theta'}^s(\{p\})}$$

for all $p \in \mathcal{P}$. We impose no further restrictions on beliefs.

Monotonic trading implies that it is easier for buyers to trade in sub-market $p' \in \mathbb{R}_+$ than in sub-market $p \in \mathbb{R}_+$ only if the opposite is true for sellers. We show how the assumption of monotonic trading is related to the existence of well-behaved matching functions for buyers and sellers when we discuss the relation of our work to Guerrieri, Shimer, and Wright (2010). One special case of monotonic trading is the case in which

$$(1 - q^s(p))(1 - q^b(p)) = 0$$

for all $p \in \mathbb{R}_+$, so that at most one side of the market is rationed in every sub-market $p \in \mathbb{R}_+$. We refer to this case as the case of *frictionless trading*. It corresponds to the case of frictionless matching in search environments.

Trivial competitive equilibria in which the set $\mathcal{P}$ is empty, and so no trade takes place in equilibrium, always exist.[5] We are interested in competitive equilibria in $\mathcal{P}$ is non-empty. It follows from monotonic trading that if $q^b(p) = q^s(p) = 0$ for some $p \in \mathbb{R}_+$, and so sub-market p is closed for trade *a priori*, then $q^s(p') > 0$ for some $p' \in \mathbb{R}_+$ only if $q^b(p') = 0$. In this case, no trade takes place in equilibrium. Indeed, if $\mathcal{P}^s = \{p \in \mathbb{R}_+ : q^s(p) > 0\}$, then market clearing implies that

$$\mu_\theta^s(\mathcal{P}^s) = \int_{\mathcal{P}_s} \beta_\theta(p) d\mu^b(p) = 0$$

for all $\theta \in \Theta$. Given this, in what follows we consider competitive equilibria in which

$$(1 - q^b(p))(1 - q^s(p)) < 1$$

[4]Indeed, $\beta_\theta(p) = \lim_{\varepsilon \to 0} f_\theta \mu_\theta^s([p - \varepsilon, p + \varepsilon]) / \overline{\mu}^s([p - \varepsilon, p + \varepsilon])$ for $\overline{\mu}^s$-almost all $p \in \mathbb{R}_+$ by (4) and the Lebesgue-Besicovitch differentiation theorem (**include reference**).

[5]An example of such an equilibrium is the list (λ, q, β, U) such that: (*i*) $\lambda^b(\{n\}) = 1$ and $\lambda_\theta^s(\{n\}) \equiv 1$; (*ii*) $q^b(p) \equiv 0$ and $q^s(p) \equiv 0$; (*iii*) $\beta_\theta(p) \equiv f_\theta$; and (*iv*) $U_\theta^s \equiv 0$ and $U^b = 0$.

for all $p \in \mathbb{R}_+$. It is straightforward to show that together with monotonic trading this last condition implies that $q^b(p) > 0$ for all $p > \inf \mathcal{P}$ when $\mathcal{P}$ is non-empty.[6]

A competitive equilibrium is *pooling* if all sellers visit the same sub-market in $\mathbb{R}_+$, that is, if there exists $p^* \in \mathbb{R}_+$ such that $\lambda^s_\theta(\{p^*\}) = 1$ for all $\theta \in \Theta$. Since $p \geq v^s_{\theta_N}$ is a necessary condition for all types of seller to be willing to trade at price p, pooling competitive equilibria can exist only if adverse selection is not severe. A competitive equilibrium is *separating* if every $p \in \mathbb{R}_+$ is in the support of at most of one the measures λ^s_θ. Thus, in a separating competitive equilibrium only one type of seller trades in each of the sub-markets that open in equilibrium.[7]

Buyer optimality implies that buyers must be indifferent between trading in all sub-markets that open in equilibrium. So, in any non-pooling competitive equilibrium the buyers' trading rule is necessarily mixed, that is, it assigns positive probability to at least two sub-markets. We say that a competitive equilibrium is *pure* if the sellers' trading rules are pure, that is, they assign probability one to a single sub-market. Otherwise, we say that a competitive equilibrium is *mixed*.

Given a competitive equilibrium, the probability of trade for the type-θ good is $\Pi_\theta = \mu^s_\theta(\mathbb{R}_+)$. We refer to the vector $\Pi = (\Pi_\theta)_{\theta\in\Theta}$ of probabilities of trade as an *allocation*. The welfare associated with the allocation Π is

$$W(\Pi) = \sum_{\theta\in\Theta} f_\theta \Pi_\theta (v^b_\theta - v^s_\theta).$$

A straightforward consequence of market clearing is that

$$W(\Pi) = U^b + \sum_{\theta\in\Theta} f_\theta U^s_\theta; \qquad (5)$$

see the Appendix for a proof. Given two competitive equilibria with allocations Π and Π', respectively, the first equilibrium is (weakly) more efficient than the second if $W(\Pi) \geq W(\Pi')$.

Since gains from trade are positive for all types of good, the first-best has all types of good trading with probability one. Rationing occurs at a competitive equilibrium if $\Pi_\theta < 1$ for at least one $\theta \in \Theta$, in which case welfare is smaller than the first-best welfare.

[6]Suppose that $\mathcal{P}$ is non-empty and let $\underline{p} = \inf \mathcal{P}$. First notice that $q^s(\underline{p}) > 0$, otherwise seller optimality implies that $q^s(p) = 0$ for all $p > \underline{p}$, which is not possible given that $\mathcal{P}$ is non-empty. Now observe that seller optimality also implies that $q^s(p)$ is nonincreasing in p for all $p \geq \underline{p}$ and is strictly decreasing in this interval as long as $q^s(p)$ is greater than zero. So, monotonic trading implies that if $q^b(p') = 0$ for some $p' > \underline{p}$ with $q^s(p') > 0$, then $q^b(p) = 0$ for all $p \in [\underline{p}, p']$, contradicting market clearing. The desired result follows from the assumption that $q^b(p) > 0$ if $q^s(p) = 0$.

[7]Not all types of seller need to trade in a separating competitive equilibrium.

2.3 Relation to Gale (1996) and Guerrieri, Shimer, and Wright (2010)

We conclude this section by discussing the relationship between our environment and the environments in Gale (1996) and Guerrieri, Shimer, and Wright (2010).

Relationship to Gale (1996)

Our environment is a version of the environment in Gale (1996) specialized to the case in which buyers and sellers trade an indivisible good. As we now discuss, there are important differences between our analysis and Gale's analysis, though.

Gale considers a general setting in which buyers and sellers trade contracts α in a convex subset A of some finite dimensional Euclidian space. The payoffs v^b to a buyer and v^s to a seller from trading a contract α depend on the seller's type θ, that is, $v^b = v^b(\alpha, \theta)$ and $v^s = v^s(\alpha, \theta)$. A seller's type is his private information and the set Θ of seller types is finite. Our setting to corresponds to the case in which $A = \mathbb{R}_+$ and a contract α specifies the exchange of an indivisible good for price $p = \alpha$. Moreover, $v^b(\alpha, \theta) = v^b_\theta - \alpha$ and $v^s(\alpha, \theta) = \alpha - v^s_\theta$.

Gale imposes the following single-crossing condition on seller payoffs. For each $\theta \in \Theta$ and $\alpha \in A$ with $v^b(\alpha, \theta) > 0$, there exists $\alpha' \in A$ with $v^s(\alpha, \theta) < v^s(\alpha', \theta)$ and $v^s(\alpha, \theta') > v^s(\alpha', \theta')$ for all $\theta \neq \theta'$.[8] This condition is a *joint* assumption on the set A of contracts and seller payoffs and it does not hold in our quasi-linear setting. Indeed, in our environment, for all $\alpha, \alpha' \in A$, $v^s(\alpha, \theta) > v^s(\alpha', \theta)$ for some $\theta \in \Theta$ if, and only if, $v^s(\alpha, \theta') > v^s(\alpha', \theta')$ for all $\theta \in \Theta$. Enriching the set of contracts to allow for random transaction prices, that is, extending A to $\Delta(\mathbb{R}_+)$, does not restore Gale's single-crossing condition.[9]

Another important difference between our analysis and Gale's analysis is that we do not impose *any* refinement on beliefs in sub-markets that do not open in equilibrium, while Gale imposes a refinement on beliefs that is analogous to the D1 refinement of Banks and Sobel (1987). In our setting, Gale's refinement would be as follows. Fix the seller payoffs $(U^s_\theta)_{\theta \in \Theta}$ and for each $p \in \mathbb{R}_+$

[8] Gale imposes a similar condition on buyer payoffs.

[9] Later in the paper we extend our analysis to the case in which buyers and sellers trade contracts specifying a probability of trade for a given price. Gale's single-crossing condition also does not hold in this more general setting.

and $\theta \in \Theta$, let $D_\theta(p)$ and $D^+_\theta(p)$ be the sets such that:

$$D_\theta(p) = \{q \in [0,1] : q(p - v^s_\theta) \geq U^s_\theta\};$$
$$D^+_\theta(p) = \{q \in [0,1] : q(p - v^s_\theta) > U^s_\theta\}.$$

By construction, $D_\theta(p)$ is the set of trading probabilities for a seller in sub-market p that make it weakly optimal for a type-θ seller to deviate his trade to p. Likewise, $D^+_\theta(p)$ is the set of trading probabilities for a seller in sub-market p that make it strictly optimal for a type-θ seller to deviate his trade to p. We say that a type-θ_i seller is more likely to deviate his trade to p than a type-θ_j seller if $D_{\theta_j}(p) \subseteq D^+_{\theta_i}(p)$.

Definition. *A competitive equilibrium is Gale-refined if for all $p \in \mathbb{R}_+$ and $\theta \in \Theta$, $\beta_\theta(p) = 0$ if other types of seller are more likely to deviate to p than type-θ sellers.*

Gale shows that his refinement and his single-crossing assumption together imply that competitive equilibria are separating and involve no rationing. In our environment, competitive equilibria need not be separating and rationing occurs in equilibrium when adverse selection is severe. We show that Gale-refined competitive equilibria are separating even if Gale's single-crossing condition does not hold. However, in our environment, separating competitive equilibria always involve rationing. In particular, such equilibria feature rationing even when adverse selection is not severe and a pooling competitive equilibrium that implements the first-best exists.

Finally, Gale restricts attention to the case of frictionless trading, while we impose the weaker requirement of monotonic trading. As we show next, assuming monotonic trading is equivalent to assuming the existence of a well-behaved matching functions for buyers and sellers.

Relationship to Guerrieri, Shimer, and Wright (2010)

Our environment is also related to the environment in Guerrieri, Shimer, and Wright (2010). They consider a setting in which buyers post contracts and sellers direct their search to a preferred contract. As is typical in the competitive search literature, Guerrieri, Shimer, and Wright take the matching functions in the market as a primitive object and define equilibria in terms of the posting decisions of buyers, the search decisions of sellers, and buyer-seller ratios.

We can define a competitive search equilibrium in our environment as follows. A contract is the price $p \in \mathbb{R}_+$ a buyer promises to pay a seller in exchange for the seller's good. To each contract p is associated a sub-market in which buyers and sellers meet in pairs to trade p. We do not make a distinction between a contract and the sub-market in which this contract is traded. Buyers have the option of not posting a contract and sellers have the option of not searching for a contract. We identify the decision of not posting a contract with the decision of posting the contract $p = n$ and the decision of not searching for a contract with the decision of searching for the contract $p = n$.

Trading rules, beliefs, and payoff vectors are as before. Given a trading rule $\lambda = (\lambda^b, (\lambda^s_\theta)_{\theta\in\Theta})$, $\lambda^b(P)$ is the probability buyers post a contract in the set P of sub-markets and $\lambda^s_\theta(P)$ is the probability a type-θ seller searches for a contract in the same set of sub-markets. A *buyer-seller ratio* is a map $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ such that $\Gamma(p)$ is the buyer-seller ratio in sub-market $p \in \mathbb{R}_+$. A *matching function* for the sellers is a nondecreasing function $m^s : \mathbb{R}_+ \cup \{\infty\} \to [0, 1]$ such that $m(\gamma)$ is the probability a seller who searches for a contract in a sub-market with buyer-seller ratio γ is matched with a buyer. A matching function for the buyers is a nonincreasing function $m^b : \mathbb{R}_+ \cup \{\infty\} \to [0, 1]$ such that $m^b(\gamma)$ is the probability a buyer who posts a contract in a sub-market with buyer-seller ratio γ is matched with a seller. We adopt the convention that $m^s(\Gamma(p)) = m^b(\Gamma(p)) = 0$ when $p = n$.

Definition. *A list $(\lambda, \Gamma, m, \beta, U)$, where λ is a trading rule, Γ is a buyer-seller ratio, $m = (m^b, m^s)$ is a pair of matching functions, β is a belief, and U is a vector of payoffs is a generalized competitive search equilibrium if it satisfies the following conditions.*

1. Buyer optimality: *For each $p \in \mathbb{R}^*$,*

$$U^b \geq m^b(\Gamma(p)) \left(\sum_\theta \beta_\theta(p) v^b_\theta - p \right), \text{ with equality holding } \lambda^b\text{-almost surely.}$$

2. Seller optimality: *For each $\theta \in \Theta$ and $p \in \mathbb{R}^*$,*

$$U^s_\theta \geq m^s(\Gamma(p))(p - v^s_\theta), \text{ with equality holding } \lambda^s_\theta\text{-almost surely.}$$

3. Market clearing: *For each $\theta \in \Theta$ and $P \subseteq \mathbb{R}_+$,*

$$\int_P \beta_\theta(p) m^b(\Gamma(p)) d\lambda^b(p) = f_\theta \int_P m^s(\Gamma(p)) d\lambda^s_\theta(p).$$

*4. **Consistency:*** $m^b(\Gamma(p)) = \Gamma(p)m^s(\Gamma(p))$ *for all* $p \in \mathbb{R}_+$.

The equilibrium notion introduced above extends the notion of a competitive search equilibrium in the sense that it takes the buyer and seller matching functions as equilibrium objects. When we take the pair (m^b, m^s) as fixed and such that $m^b(\gamma) = \gamma m^s(\gamma)$ for all $\gamma \in \mathbb{R}_+ \cup \{\infty\}$, this equilibrium notion reduces to a standard competitive search equilibrium with adverse selection. We claim that competitive equilibria and generalized competitive search equilibria coincide in our environment.

Suppose first that $(\lambda, \Gamma, m, \beta, U)$ is a generalized competitive search equilibrium and define the trading outcome $q = (q^b, q^s)$ as follows: $q^b(p) = m^b(\Gamma(p))$ and $q^s(p) = m^s(\Gamma(p))$. Since $m^s(\gamma)$ is nondecreasing in γ, $q^s(p') < q^s(p)$ only if $\Gamma(p') < \Gamma(p)$. This, in turn, implies that $q^b(p') \geq q^s(p)$, as $m^b(\gamma)$ is nonincreasing in γ. Thus, (λ, q, β, U) is a competitive equilibrium. Suppose now that (λ, q, β, U) is a competitive equilibrium and set $\overline{\lambda}_s = \sum_{\theta \in \Theta} f_\theta \lambda^s_\theta$. By construction, $\overline{\lambda}^s(P)$ is the mass of sellers who visit a sub-market in $P \subseteq \mathbb{R}^*$. In the Appendix, we establish that

$$\frac{q^s(p)}{q^b(p)} = \lim_{\varepsilon \to 0} \frac{\lambda^b([p-\varepsilon, p+\varepsilon])}{\overline{\lambda}^s([p-\varepsilon, p+\varepsilon])}$$

for $\overline{\lambda}^s$-almost all $p \in \mathbb{R}_+$. This justifies interpreting the ratio $q^s(p)/q^b(p)$ as the buyer-seller ratio in sub-market p. Let then $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{\infty\}$ be such that $\Gamma(p) = q^s(p)/q^b(p)$. We also show in the Appendix that monotonic trading implies that there exist matching functions m^s for the sellers and m^b for the buyers such that $m^s(\Gamma(p)) = q^s(p)$ and $m^b(\Gamma(p)) = q^b(p)$ for all $p \in \mathbb{R}_+$.[10] From this, it follows that $(\lambda, \Gamma, m, \beta, U)$ is a generalized competitive search equilibrium.

Similary to Gale, Guerrieri, Shimer, and Wright impose a single-crossing condition on seller payoffs. This single-crossing condition differs from Gale's condition in that it is a local condition: for any type of seller and any contract, there exists an arbitrarily close contract that makes this type of seller strictly better off and all lower types of seller strictly worse off. As is the case with Gale's condition, this local single-crossing condition does not hold in our quasi-linear setting.

Guerrieri, Shimer, and Wright also impose a restriction on contracts that are not posted in equilibrium. Using the equivalence between generalized competitive search equilibria and competitive equilibria, their restriction amounts to the following restriction on competitive equilibria.

[10]When we impose the additional restriction that trading is frictionless, $m(\gamma) = \min\{\gamma, 1\}$.

Definition. *A competitive equilibrium is GSW-refined if for all* $p \in \mathbb{R}_+$ *with* $q^b(p) > 0$ *we have that: (i)* $\beta_\theta(p) > 0$ *implies that* $U^s_\theta = q^s(p)(p - v^s_\theta)$*; and (ii)* $p < v^s_\theta$ *implies that* $\beta_\theta(p) = 0$.

Condition (*i*) implies that if a buyer posts an out of equilibrium contract that attracts sellers ($q^b(p) > 0$), then all types of seller that find this contract attractive are indifferent between it and the contracts that they search for in equilibrium. This captures the idea that if trading some out of equilibrium contract p were profitable for at least one type θ of seller, then type-θ sellers would line up in the sub-market where p is traded up to the point where the probability that they trade p is low enough to make them indifferent between searching for p and searching for an equilibrium contract. Condition (*i*) extends to an adverse selection setting the market utility property of competitive search models without adverse selection. We show in the next section that it implies that competitive equilibria are separating, even when adverse selection is not severe.

Condition (*ii*) imposes the reasonable requirement that if a buyer posts an out of equilibrium contract that is not profitable to type-θ sellers, then it does not attract such sellers. Since $q^b(p) > 0$ for all $p > \underline{p} = \inf \mathcal{P}$, this second condition implies that for each $\theta \in \Theta$, the belief β is such that $\beta_\theta(p) = 0$ for all $p \in (\underline{p}, v^s_\theta)$.

3 Basic Results

In this section, we establish a number of results about competitive equilibria. We first discuss equilibrium existence. We show that pooling competitive equilibria exist if, and only if, adverse selection is not severe and separating competitive equilibria always exist.

We then establish a key fact about the behavior of sellers in any competitive equilibrium, namely, that a seller of a given type never trades at a higher price than a seller of a higher type and that if a seller of a given type does not trade in equilibrium, then higher type sellers also do not trade in equilibrium. A consequence of this result is that rationing occurs in any non-pooling competitive equilibrium. In particular, rationing occurs in equilibrium if adverse selection is severe.

After that, we establish some results about the number of sub-markets that open in equilibrium. In particular, we show that as far as allocations are concerned, restricting attention to competitive equilibria in which only a finite number of sub-markets open in equilibrium is without loss of generality. This result plays an important role when we discuss equilibrium welfare.

The next set of results we obtain concern welfare properties of competitive equilibria. It follows from these results that restricting attention to pure competitive equilibria in which the buyer's payoff is zero is without loss of generality as far as market efficiency is concerned. This fact is useful when we discuss constrained (in)efficiency of competitive equilibria.

We conclude this section by showing that the refinements by Gale and Guerrieri, Shimer, and Wright imply that competitive equilibria are separating.

In what follows, we let $supp[\nu]$ denote the support of a measure ν on $\mathbb{R}^*$. Moreover, for each $\theta \in \Theta$, we let $\mathcal{P}_\theta = supp[\mu^s_\theta]$, $\underline{p}_\theta = \inf \mathcal{P}_\theta$, and $\overline{p}_\theta = \sup \mathcal{P}_\theta$, where $\underline{p}_\theta = n$ if $\mathcal{P}_\theta = \emptyset$.

3.1 Existence

First, notice that a pooling competitive equilibrium exists if adverse selection is not severe. Indeed, suppose that $\overline{v}^b = \sum_{\theta \in \Theta} f_\theta v^b_\theta \geq v^s_{\theta_N}$ and let $p^* \in [v^s_{\theta_N}, \overline{v}^b]$. Consider the trading rule λ such that $\lambda^s_\theta(\{p^*\}) = \lambda^b(\{p^*\}) = 1$ for all $\theta \in \Theta$. Let the trading outcome q be such that: (*i*) $q^b(p) = 1$ if $p \geq p^*$ and $q^b(p) = 0$ otherwise; and (*ii*) $q^s(p) = 1$ if $p \leq p^*$ and $q^s(p) = 0$ otherwise. Notice that trading is frictionless. Now let the belief β be such that $\beta_\theta(p) \equiv f_\theta$ for all $\theta \in \Theta$ and the vector of payoffs U be such that $U^b = \overline{v}^b - p^*$ and $U^s_\theta = p^* - v^s_\theta$ for all $\theta \in \Theta$. Clearly, buyer and seller optimality hold. Market clearing also holds. Thus, (λ, q, β, U) is a competitive equilibrium. Notice that (λ, q, β, U) implements the first-best.

We now show that a separating competitive equilibrium always exists. Our candidate equilibrium is as follows. The trading rule λ is such that $\lambda^s_\theta(\{v^b_\theta\}) = 1$ for all $\theta \in \Theta$; we define λ^b below. The trading outcome for buyers is q^b such that $q^b(p) = 0$ if $p < v^b_{\theta_1}$ and $q^b(p) = 1$ otherwise. The trading outcome q^s for sellers is such that $q^s(p) = 1$ if $p \leq v^b_{\theta_1}$, $q^s(p) = q^s(v^b_{\theta_i})$ if $p \in (v^b_{\theta_{i-1}}, v^b_{\theta_i})$, $q^s(p) = 0$ if $p > v^b_{\theta_N}$, where the probabilities $q^s(v^b_{\theta_2})$ to $q^s(v^b_{\theta_N})$ are defined recursively as follows:

$$q^s(v^b_{\theta_{i+1}}) = q^s(v^b_{\theta_i}) \cdot \frac{v^b_{\theta_i} - v^s_{\theta_i}}{v^b_{\theta_{i+1}} - v^s_{\theta_i}}$$

for all $i \in \{1, \ldots, N-1\}$. To complete the definition of λ, let λ^b be such that $\lambda^b(\{v^b_\theta\}) = q^s(v^b_\theta)$ for all $\theta \in \Theta$ and define $\lambda^b(\{n\})$ residually as $\lambda^b(\{n\}) = 1 - \sum_{\theta \in \Theta} q^s(v^b_\theta)$. By construction, trading is frictionless and $\mathcal{P} = \{v^b_{\theta_1}, \ldots, v^b_{\theta_N}\}$. Finally, let the belief β be such that $\beta_\theta(v^b_\theta) = 1$ for all $\theta \in \Theta$ and the vector of payoffs U be such that $U^b = 0$ and $U^s_\theta = q^s(v^b_\theta)(v^b_\theta - v^s_\theta) > 0$ for all $\theta \in \Theta$; we complete the description of β below.

We claim that (λ, q, β, U) is a competitive equilibrium. It is immediate to see that market clearing holds.[11] Given that the buyers's expected payoff is zero in the sub-markets that open in equilibrium, buyer optimality holds as long as β is such that the buyers' expected payoff from trading in any sub-market that does not open in equilibrium is nonpositive. This is the case if, for instance, $\beta_{\theta_1}(p) = 1$ for all $p < v^b_{\theta_2}$, $\beta_{\theta_i}(p) = 1$ for all $p \in (v^b_{\theta_i}, v^b_{\theta_{i+1}})$ and all $i \in \{1, \ldots, N-1\}$, and $\beta_{\theta_N}(p) = 1$ if $p > v^b_{\theta_N}$. In the Appendix we show that seller optimality holds as well.

Proposition 1 summarizes the discussion about equilibrium existence.

Proposition 1. *A pooling competitive equilibrium exists if, and only if, adverse selection is not severe. A separating competitive equilibrium always exists.*

The buyers' payoff is zero in the separating competitive equilibrium that we constructed. It is possible to construct separating competitive equilibria in which the buyers's payoff is positive. In the next section we show that when trading is frictionless, a mild refinement on beliefs implies that buyers obtain zero payoff in any separating equilibrium.

3.2 Monotonicity

We now establish a key fact about the behavior of sellers in any competitive equilibrium. For ease of exposition, we adopt the convention that $p < n$ for all $p \in \mathbb{R}_+$.

Proposition 2. *Consider a competitive equilibrium and let $i > j$. Then $p_i \geq p_j$ for all $p_i \in \mathcal{P}_{\theta_i}$ and $p_j \in supp[\lambda^s_{\theta_j}]$. Moreover, $\mathcal{P}_{\theta_j} = \emptyset$ implies that $\mathcal{P}_{\theta_i} = \emptyset$.*

Since, by seller optimality, $q^s(p) = 0$ implies that $q^s(p') = 0$ for all $p > p'$, Proposition 2 implies that if $\mathcal{P}_{\theta_i} \neq \emptyset$ for some $i \in \{1, \ldots, N\}$, then $\mathcal{P}_{\theta_j} \neq \emptyset$ for all $j < i$. Thus, if a type of seller trades with positive probability in equilibrium, then every lower type of seller trades with probability one in equilibrium.

[11] Indeed, $\beta_\theta(v^b_\theta) q^b(v^b_\theta) \lambda^b(\{v^b_\theta\}) = q^s(v^b_\theta) = q^s(v^b_\theta) \lambda^s_\theta(\{v^b_\theta\})$ for all $\theta \in \Theta$.

Here, we prove Proposition 2 in the case in which the measures λ^s_θ have finite support. In the Appendix we show that the result holds in general. Let $i > j$ and suppose that $p_i \in \mathcal{P}_{\theta_i}$ and $p_j \in supp[\lambda^s_{\theta_j}]$, so that $q^s(p_i) > 0$ and $p_i \in \mathbb{R}_+$ *a fortiori*. Seller optimality implies that:

$$U^s_{\theta_i} = q^s(p_i)(p_i - v^s_{\theta_i}) \geq q^s(p_j)(p_j - v^s_{\theta_i});$$
$$U^s_{\theta_j} = q^s(p_j)(p_j - v^s_{\theta_j}) \geq q^s(p_i)(p_i - v^s_{\theta_j}).$$

Summing the two inequalities, we obtain

$$(q^s(p_i) - q^s(p_j))(v^s_{\theta_i} - v^s_{\theta_j}) \geq 0.$$

Since $v^s_{\theta_i} > v^s_{\theta_j}$, we then have that $q^s(p_j) \geq q^s(p_i)$. There are two cases to consider: $p_j > v^s_{\theta_i}$ and $p_j \leq v^s_{\theta_i}$. Given that $p_i \geq v^s_{\theta_i}$ by seller optimality, it follows that $p_i \geq p_j$ if $p_j \leq v^s_{\theta_i}$. Suppose then that $p_j > v^s_{\theta_i}$. Since $q^s(p_i) > 0$, we then have that

$$q^s(p_i)(p_i - v^s_{\theta_i}) \geq q^s(p_i)(p_j - v^s_{\theta_i}) > 0,$$

and so $p_i \geq p_j$ as well.

To conclude the proof of Proposition 2, suppose that $\mathcal{P}_{\theta_j} = \emptyset$, so that type-$\theta_j$ sellers do not trade in equilibrium. Then $U^s_{\theta_j} = 0$, and so seller optimality implies that

$$0 \geq q^s(p_i)(p_i - v^s_{\theta_j})$$

for all $p_i \geq v^s_{\theta_i}$. Given that $v^s_{\theta_i} > v^s_{\theta_j}$, we then have that $q^s(p_i) = 0$ for all $p_i \geq v^s_{\theta_i}$. This implies that $\mathcal{P}_{\theta_i} = \emptyset$, as no type-$\theta_i$ seller would direct his trade to a sub-market $p < v^s_{\theta_i}$ if $q^s(p) > 0$.

It follows from Proposition 2 that the sets of prices at which two different types of seller can trade have at most one point in common. Morever, $\mathcal{P}_{\theta_1} \neq \emptyset$, otherwise no trade would take place in equilibrium. Another consequence of Proposition 2 is that the probabilities of trade Π_θ are nonincreasing in θ and $\Pi_{\theta_1} > \Pi_{\theta_N}$ in any non-pooling competitive equilibrium. Hence, rationing occurs in any non-pooling competitive equilibrium. In particular, all separating competitive equilibria involve rationing. The proof of Corollary 1 is in the Appendix.

Corollary 1. Π_θ *is nonincreasing in* θ *in any competitive equilibrium and* $\Pi_{\theta_1} > \Pi_{\theta_N}$ *in any non-pooling competitive equilibrium.*

3.3 Equilibrium Prices

The next result we establish concerns the number of sub-markets that open in equilibrium. We say that two competitive equilibria are outcome equivalent if they have the same allocations. Recall that $|A|$ denotes the cardinality of a set A.

Lemma 1. *Consider a competitive equilibrium. If $U^b = 0$, then $|\mathcal{P}_\theta| \leq 3$ for all $\theta \in \Theta$. Moreover, if there exists $\theta \in \Theta$ such that $|\mathcal{P}_\theta| \geq 4$, then there exists an outcome equivalent competitive equilibrium in which $|\mathcal{P}_\theta| \leq 3$ for all $\theta \in \Theta$.*

The proof of Lemma 1 is in the Appendix. We provide a sketch of the proof in what follows. Consider first a competitive equilibrium in which $U^b = 0$ and suppose, by contradiction, that there exists $\theta \in \mathcal{P}_\theta$ such that $\mathcal{P}_\theta$ has four or more elements. By assumption, there exists $p, p' \in \mathcal{P}_\theta$ with $\underline{p}_\theta < p < p' < \overline{p}_\theta$. Moreover, by market clearing and buyer optimality, we can take p and p' to be such that both $q^b(p)$ and $q^b(p')$ are positive and

$$U^b = q^b(p)\left(\sum_{\theta\in\Theta}\beta_\theta(p)v_\theta^b - p\right) = q^b(p')\left(\sum_{\theta\in\Theta}\beta_\theta(p')v_\theta^b - p'\right).$$

Now observe by Proposition 2 that only type-θ sellers visit the sub-markets p and p'. Bayes' rule then implies that $\beta_\theta(p) = \beta_\theta(p') = 1$. Given that $U^b = 0$ and both $q^b(p)$ and $q^b(p')$ are positive, we can then conclude that $p = v_\theta^b = p'$, a contradiction.

Now consider a competitive equilibrium in which there exists $\theta \in \Theta$ such that $\mathcal{P}_\theta$ has at least four elements. Let $\mathcal{P}_\theta^0 = \mathcal{P}_\theta \setminus \{\underline{p}_\theta, \overline{p}_\theta\}$. We show in the Appendix that since type-θ sellers are the only sellers who visit the sub-markets in $\mathcal{P}_\theta^0$, we can construct a competitive equilibrium in which we replace all sub-markets in $\mathcal{P}_\theta^0$ by a single sub-market p_θ^0 in the interval $(\underline{p}_\theta, \overline{p}_\theta)$ in such a way that type-θ sellers obtain the same payoff visiting p_θ^0 that they obtain visiting the sub-markets in $\mathcal{P}_\theta^0$ in the original equilibrium. It follows from this that the probability of trade for the type-θ good is the same in the new equilibrium as in the original equilibrium.

Given that only type-θ_1 sellers visit sub-market $\underline{p}_{\theta_1}$ if $\mathcal{P}_{\theta_1}$ has at least two elements and only type-θ_N sellers visit sub-market $\overline{p}_{\theta_N}$ if $\mathcal{P}_{\theta_N}$ has at least two elements, the same argument as above shows that the following result holds. Corollary 2 is useful when we analyze the two-type case.

Corollary 2. *Consider a competitive equilibrium. If* $U^b = 0$*, then* $|\mathcal{P}_{\theta_1}| \leq 2$ *and* $|\mathcal{P}_{\theta_N}| \leq 2$*. Moreover, if either* $|\mathcal{P}_{\theta_1}| \geq 3$ *or* $|\mathcal{P}_{\theta_N}| \geq 3$*, then there exists an outcome equivalent competitive equilibrium in which* $|\mathcal{P}_{\theta_1}| \leq 2$ *and* $|\mathcal{P}_{\theta_N}| \leq 2$*.*

A consequence of Lemma 1 is that we can restrict attention to competitive equilibria in which a finite number of sub-markets open in equilibrium when considering the welfare properties of competitive equilibria. We make use of this fact in what follows.

3.4 Welfare

The next two results we establish concern welfare properties of competitive equilibria. The first result shows that competitive equilibria in which the buyers' payoff is positive are weakly inefficient. The second result shows that mixed competitive equilibria are also weakly inefficient.

Lemma 2. *For any competitive equilibrium with* $U^b > 0$ *there exists a more efficient competitive equilibrium with* $U^b = 0$*.*

The proof of Lemma 2 is in the Appendix. In what follows, we provide a sketch of the proof. Consider a competitive equilibrium (λ, q, β, U) with $U^b > 0$. We know from Lemma 1 that we can assume that the set $\mathcal{P}$ of sub-markets that open in equilibrium is finite. Let $\mathcal{P} = \{p_1, \ldots, p_M\}$, with $M \geq 1$, and to each $p_j \in \mathcal{P}$, let $\Theta_j = \{\theta \in \Theta : \lambda^s_\theta(\{p_j\}) > 0\}$ be the set of types of seller who direct their trade to sub-market p_j. Moreover, let $\eta(\theta, p_j) = \lambda^s_\theta(\{p_j\})$ be the mass of type-θ sellers who direct their trade to sub-market p_j and

$$\widehat{p}_j = \frac{1}{\displaystyle\sum_{\theta \in \Theta_j} \eta(\theta, p_j)} \sum_{\theta \in \Theta_j} \eta(\theta, p_j) v^b_\theta.$$

By Bayes' rule, the expected payoff to a buyer who directs his trade to sub-market p_j is $U^b = q^b(p_j)(\hat{p}_j - p_j)$. Given that $q^b(p_j) > 0$ for all $j \in \{1, \ldots, M\}$, as $U^b > 0$, it follows from buyer optimality that for each $j \in \{1, \ldots, M\}$,

$$p_j = \widehat{p}_j - \frac{U^b}{q^b(p_j)} < \hat{p}_j.$$

Now let $\hat{\lambda}$ be the trading rule such that for each $j \in \{1, \ldots, M\}$, $\hat{\lambda}^s_\theta(\{\hat{p}_j\}) = \lambda^s_\theta(p_j)$ for all $\theta \in \Theta_j$. By construction, $\hat{\lambda}$ is such that a type-θ seller who directs his trade to p_j with positive probability under λ directs his trade to $\hat{p}_j$ with the same probability under $\hat{\lambda}$. In the Appendix we show that there exists a trading outcome $\hat{q}$ with the property that $\hat{q}^s(\hat{p}_j) \geq q^s(p_j)$ for all $j \in \{1, \ldots, M\}$, a belief system $\hat{\beta}$, and a vector of payoffs $\hat{U}$ such that $(\hat{\lambda}, \hat{q}, \hat{\beta}, \hat{U})$ is a competitive equilibrium. Notice that $\hat{U}^b = 0$. Welfare in $(\hat{\lambda}, \hat{q}, \hat{\beta}, \hat{U})$ is weakly higher than welfare in (λ, q, β, U) as the probability of trade of each type of good is weakly higher in the first equilibrium.

The intuition for this result is that by raising the prices at which trade takes place one relaxes the sellers' incentive compatibility constraints, allowing for (weakly) higher probabilities of trade. Indeed, a type-θ seller with $\theta \in \Theta_j$ has no incentive to behave as if his type is in Θ_{j+1} if

$$q^s(p_j)(p_j - v^s_\theta) \geq q^s(p_{j+1})(p_{j+1} - v^s_\theta). \tag{6}$$

Since $q^s(p_j) > q^s(p_{j+1})$ by seller optimality, monotonic trading implies that $q^b(p_j) \leq q^b(p_{j+1})$. Thus, given that $\hat{p}_j - p_j \geq \hat{p}_{j+1} - p_j$, one relaxes the (sufficient) local incentive compatibility constraint (6) by raising p_j to $\hat{p}_j$. This, in turn, allows one to increase the probabilities for trade for all goods of type $\theta \in \bigcup_{j=2}^M \Theta_j$.

Lemma 2 shows that as far as welfare is concerned, we can restrict attention to competitive equilibria in which the buyers' payoff is zero. The next result shows that we can also restrict attention to pure competitive equilibria.

Lemma 3. *For any mixed competitive equilibrium with $U^b = 0$ there exists a more efficient pure competitive equilibrium.*

The proof of Lemma 3 is in the Appendix. A sketch of the proof is as follows. Let (λ, q, β, U) be a mixed competitive equilibrium in which the buyers' payoff is zero. As in the proof of the previous lemma, we can assume that only a finite number of sub-markets open in equilibrium. Let $\mathcal{P} = \{p_1, \ldots, p_M\}$, with $M \geq 1$, be the set of sub-markets that open in equilibrium. Since $U^b = 0$, buyer optimality implies that

$$p_j = \sum_{\theta \in \Theta} \beta_\theta(p_j) v^b_\theta.$$

Now let $\hat{\Theta}_j = \{\theta \in \Theta : \underline{p}_\theta = p_j\}$, with $j \in \{1, \ldots, M\}$, be the set of seller types for which p_j is the smallest price at which they trade in equilibrium. Moreover, for each $j \in \{1, \ldots, M\}$, let

$$\hat{p}_j = \frac{1}{\sum\limits_{\theta \in \widehat{\Theta}_j} f_\theta} \sum_{\theta \in \widehat{\Theta}_j} f_\theta v^b_\theta.$$

Notice that $\hat{p}_j \geq p_j$ for all $j \in \{1, \ldots, M\}$ as Proposition 2 implies that all but the highest type in $\hat{\Theta}_j$ chooses p_j with probability one. Consider then the trading rule $\hat{\lambda}$ such that a type-θ seller chooses $\hat{p}_j$ if $\theta \in \hat{\Theta}_j$. An argument similar to the one used in the proof of the previous lemma shows that we can construct a list $(\hat{q}, \hat{\beta}, \hat{U})$ such that $(\hat{\lambda}, \hat{q}, \hat{\beta}, \hat{U})$ is a pure competitive equilibrium that is weakly more efficient than the original competitive equilibrium; as before, raising the prices at which trade takes place relaxes the sellers' incentive compatibility constraints, allowing for a (weakly) higher probability of trade in all sub-markets that open in equilibrium.

Lemmas 2 and 3 imply that for any competitive equilibrium there exists a (weakly) more efficient pure competitive equilibrium in which the buyers' payoff is zero. Proposition 3 summarizes our results about equilibrium welfare.

Proposition 3. *For any competitive equilibrium there exists a more efficient pure competitive equilibrium with $U^b = 0$.*

3.5 Refined Equilibria

We conclude this section by discussing how the refinements in Gale (1996) and Guerrieri, Shimer, and Wright (2010) restrict the set of competitive equilibria. We first show that Gale's refinement implies that every competitive equilibrium is separating.

Proposition 4. *Every Gale-refined competitive equilibrium is separating.*

Here, we prove Proposition 4 in the case in which the set of sub-markets that open in equilibrium is finite. In the Appendix we show that this result holds in general. Consider a non-separating competitive equilibrium. By assumption, there exists $p \in \mathbb{R}_+$ with $q^s(p) > 0$ and $q^b(p) > 0$ such that $\lambda^s_\theta(\{p\}) > 0$ for at least two types of seller. Let θ_i be the highest type of seller that visits sub-market p; notice that $i > 1$. By Bayes' rule, $\sum_{\theta\in\Theta}\beta_\theta(p)v^b_\theta < v^b_{\theta_i}$. Since, by buyer optimality,

$$U^b = q^b(p)\left(\sum_{\theta\in\Theta}\beta_\theta(p)v^b_\theta - p\right)$$

and $U^b \geq 0$, we then have that $p < v^b_{\theta_i}$.

We claim that type-θ_i sellers are more likely to deviate to any $p' \in (p, v^b_{\theta_i})$ than type-θ_j sellers if $j < i$. Let $p' \in (p, v^b_{\theta_i})$, $j < i$, and suppose that $q(p' - v^s_{\theta_j}) \geq q^s(p)(p - v^s_{\theta_j})$. If $q < q^s(p)$, then

$$\begin{aligned} q(p' - v^s_{\theta_i}) &= q(p' - v^s_{\theta_j}) + q(v^s_{\theta_j} - v^s_{\theta_i}) \\ &\geq q^s(p)(p - v^s_{\theta_i}) + (q^s(p) - q)(v^s_{\theta_i} - v^s_{\theta_j}) \\ &> q^s(p)(p - v^s_{\theta_i}). \end{aligned}$$

On the other hand, if $q \geq q^s(p)$, then $q(p' - v^s_{\theta_i}) > q^s(p)(p - v^s_{\theta_i})$ as well. So, $q \in D_{\theta_j}(p')$ implies that $q \in D^+_{\theta_i}(p')$, which establishes the desired result.

Now observe that since $q^s(p') < q^s(p)$ for all $p' > p$ by seller optimality, monotonic trading implies that $q^b(p') \geq q^b(p)$ for all $p' > p$. On the other hand, buyer optimality implies that

$$q^b(p)\left(\sum_{\theta\in\Theta}\beta_\theta(p)v^b_\theta - p\right) \geq q^b(p')\left(\sum_{\theta\in\Theta}\beta_\theta(p')v^b_\theta - p'\right)$$

for all $p > p'$. Thus, there exists $p' \in (p, v^b_{\theta_i})$ such that

$$\sum_{\theta\in\Theta}\beta_\theta(p')v^b_\theta \leq \sum_{\theta\in\Theta}\beta_\theta(p)v^b_\theta.$$

This, in turn, is only possible if there exists $j < i$ such that $\beta_{\theta_j}(p') > 0$, as type-θ_i sellers are the highest type of seller that visits sub-market p. So, the equilibrium under consideration is not refined. This concludes the proof.

We now show that condition (*i*) in the refinement by Guerrieri, Shimer, and Wright also implies that every competitive equilibrium is separating.

Proposition 5. *Every GSW-refined competitive equilibrium is separating.*

For ease of exposition, we prove Proposition 5 under the assumption that the set of sub-markets that open in equilibrium is finite. In the Appendix we show that this result holds in general.

Consider a non-separating competitive equilibrium. By assumption, there exist $j, k \in \{1, \ldots, N\}$ with $j < k$ and $p \in \mathbb{R}_+$ with $q^s(p) > 0$ and $q^b(p) > 0$ such that $p \in \mathcal{P}_{\theta_j} \cap \mathcal{P}_{\theta_k}$. Assume, without loss, that θ_k is the highest type of seller that visits sub-market p. Since $q^s(p) > q^s(p')$ for all $p' > p$ by seller optimality, monotonic trading implies that $q^b(p') \geq q^b(p) > 0$ for all $p' > p$. Thus, given that buyer optimality implies that

$$U^b \geq q^b(p') \left(\sum_{\theta \in \Theta} \beta_\theta(p') v_\theta^b - p' \right)$$

for all $p' > p$, it must be that $\beta_{\theta_i}(p') > 0$ for some $i < k$ if $p' > p$ is sufficiently close to p. Otherwise, there exists $p' > p$ such that

$$\sum_{\theta \in \Theta} \beta_\theta(p') v_\theta^b - p' \geq v_{\theta_k}^b - p' > \sum_{\theta \in \Theta} \beta_\theta(p) v_\theta^b - p,$$

contradicting buyer optimality; the strict inequality follows from the fact that $\beta_{\theta_j}(p) > 0$.

Let $p' > p$ and $i < k$ be such that $\beta_{\theta_i}(p') > 0$. If $q^s(p')(p' - v_{\theta_i}^s) \geq q^s(p)(p - v_{\theta_i}^s)$, then

$$\begin{aligned} q^s(p')(p' - v_{\theta_k}^s) &= q^s(p')(p' - v_{\theta_i}^s) + q^s(p')(v_{\theta_i}^s - v_{\theta_k}^s) \\ &\geq q^s(p)(p - v_{\theta_i}^s) + q^s(p')(v_{\theta_i}^s - v_{\theta_k}^s) \\ &> q^s(p)(p - v_{\theta_k}^s), \end{aligned}$$

which violates seller optimality; the strict inequality follows from the fact that $q^s(p) > q^s(p')$. Consequently, $q^s(p')(p' - v_{\theta_i}^s) < q^s(p)(p - v_{\theta_i}^s) \leq U_{\theta_i}^s$ and the equilibrium under consideration is not GSW-refined.

4 Further Results

In this section, we establish some further results about competitive equilibria. First, we analyze the case of frictionless trading. Then, we provide a complete characterization of competitive equilibrium allocations in the two-type when adverse selection is severe.

4.1 Frictionless Trading

Consider the case of frictionless trading. We first show that we can strengthen Lemma 1 to prove that only a finite number of sub-markets open in any competitive equilibrium. We then show that a mild refinement of beliefs in sub-markets that do not open in equilibrium implies that the buyers' payoff is zero in any non-pooling competitive equilibrium. We also show that the buyers' payoff is zero in any separating competitive equilibrium and provide conditions under which the buyers' payoff is zero in any non-pooling competitive equilibrium without having to impose any refinement on beliefs. These conditions are automatically satisfied in the two-type case.

Proposition 6. *Only a finite number of sub-markets open in equilibrium if trading is frictionless.*

The proof of Proposition 6 is in the Appendix. A sketch of the proof is as follows. First notice that seller optimality implies that $q^s(p) < 1$ for all $p > \underline{p}_{\theta_1}$, otherwise type-$\theta_1$ sellers' behavior would not be optimal. Frictionless trading then implies that $q^b(p) = 1$ for all $p > \underline{p}_{\theta_1}$. The argument is now similar to the proof of Lemma 1. Suppose there exists $\theta \in \Theta$ such that $\mathcal{P}_\theta$ has at least four elements. Then there exist $p, p' \in \mathcal{P}_\theta$ with $\underline{p}_\theta < p < p' < \overline{p}_\theta$. Since, by Proposition 2, only type-θ sellers visit sub-markets p and p', Bayes' rule and buyer optimality imply that $v^b_\theta - p = v^b_\theta - p'$, a contradiction.

We say that the belief β is *monotonic* if the expected value of the good to buyers is nondecreasing in its price, that is, if $\mathcal{E}^b(p) = \sum_{\theta\in\Theta} \beta_\theta(p) v^b_\theta$ is nondecreasing in p. Proposition 2 implies $\mathcal{E}^b(p)$ is nondecreasing in p when p lies in the set prices at which trade takes place in equilibrium. Monotonicity of beliefs extends this property of $\mathcal{E}^b(p)$ to all possible prices, and is a reasonable requirement. It is satisfied, for instance, in Gale-refined and GSW-refined competitive equilibria; see the Appendix for a proof. The next result shows that when matching is frictionless, the payoff to buyers is zero in any competitive equilibrium in which beliefs are monotonic.

Proposition 7. *Suppose beliefs are monotonic. The payoff to buyers is zero in any non-pooling competitive equilibrium if trading is frictionless.*

A sketch of the proof of Proposition 7 is as follows; see the Appendix for details. Consider a non-pooling competitive equilibrium with $U^b > 0$. Since, by seller optimality, $q^s(p) < q^s(\underline{p}_{\theta_1}) \leq 1$ for all $p > \underline{p}_{\theta_1}$, frictionless trading implies that $q^b(p) = 1$ for all $p > \underline{p}_{\theta_1}$. Then $q^b(\underline{p}_{\theta_1}) < 1$, otherwise market clearing would be violated. Indeed, since $U^b > 0$, and so no buyer visits sub-market n, $q^b(\underline{p}_{\theta_1}) = 1$ would imply that all buyers trade in equilibrium. However, $q^s(p) < 1$ for all $p > \underline{p}_{\theta_1}$ implies that not all sellers trade in equilibrium, as the equilibrium is non-pooling, a contradiction. Since, by buyer optimality,

$$U^b = q^b(\underline{p}_{\theta_1})(\mathcal{E}^b(\underline{p}_{\theta_1}) - \underline{p}_{\theta_1}) \geq \mathcal{E}^b(p) - p$$

for all $p > \underline{p}_{\theta_1}$, we can then conclude that $\mathcal{E}^b(p) < \mathcal{E}^b(\underline{p}_{\theta_1})$ if $p > \underline{p}_{\theta_1}$ is sufficiently close to $\underline{p}_{\theta_1}$. Thus, β is not monotonic. The desired result follows from taking the contrapositive.

Given that $\mathcal{E}^b(p) \geq v^b_{\theta_1}$ for all $p \in \mathbb{R}_+$, it follows from the proof of Proposition 7 that when trading is frictionless, a non-pooling competitive equilibrium in which the buyers' payoff is positive is possible only if $\mathcal{E}^b(\underline{p}_{\theta_1}) > v^b_{\theta_1}$. This, in turn, is possible only if sellers of higher type pool with type-θ_1 sellers in sub-market $\underline{p}_{\theta_1}$. In particular, the equilibrium cannot be separating.

Consider a competitive equilibrium in which sellers of higher type pool with type-θ_1 sellers in $\underline{p}_{\theta_1}$ and let θ_k, with $k > 1$, be highest type of seller that directs his trade (with positive probability) to $\underline{p}_{\theta_1}$. By Proposition 2, the type-θ_j sellers with $j < k$ direct their trade with probability one to $\underline{p}_{\theta_1}$, and so Bayes' rule implies that

$$\mathcal{E}(\underline{p}_{\theta_1}) \leq \frac{1}{\sum_{i=1}^k f_{\theta_i}} \sum_{i=1}^k f_{\theta_i} v^b_{\theta_i}.$$

On the other hand, $\underline{p}_{\theta_1} \geq v^s_{\theta_k}$ by seller optimality. Hence, a necessary condition for the type of equilibrium under consideration to exist is that $\sum_{i=1}^k f_{\theta_i} v^b_{\theta_i} \geq (\sum_{i=1}^k f_{\theta_i}) v^s_{\theta_k}$.

We say that adverse selection is *extreme* if

$$\sum_{i=1}^k f_{\theta_i} v^b_{\theta_i} < \left(\sum_{i=1}^k f_{\theta_i} \right) v^s_{\theta_k}$$

for all $k \in \{2, \ldots, N\}$. Extreme adverse selection implies that for all $k \in \{2, \ldots, N\}$, there exists no price at which the first k types of good can trade at this price. Notice that extreme adverse selection implies severe adverse selection and that the two conditions coincide when there are only two types of good in the market. The next result follows from the preceding discussion

Proposition 8. *If trading is frictionless, then $U^b = 0$ in every separating competitive equilibrium. Moreover, $U^b = 0$ in any non-pooling competitive equilibrium if adverse selection is extreme.*

It follows immediately from Proposition 8 that if trading is frictionless, then every separating competitive equilibrium is such that $\mathcal{P}_\theta = \{v^b_\theta\}$ for all $\theta \in \Theta$ for which $\mathcal{P}_\theta$ is non-empty.

4.2 Two-Type Case with Severe Adverse Selection

Suppose that $\Theta = \{\theta_1, \theta_2\}$ and assume that adverse selection is severe.

Consider first the case of frictionless trading. Then only non-pooling competitive equilibria exist and $U^b = 0$ in all such equilibria. By buyer optimality, $\underline{p}_{\theta_1} = \mathcal{E}^b(\underline{p}_{\theta_1})$. On the other hand, $\mathcal{E}^b(\underline{p}_{\theta_1}) < \sum_{\theta\in\Theta} f_\theta v^b_\theta$ by Proposition 2, Bayes' rule, and the fact that equilibria are non-pooling. Hence, $\underline{p}_{\theta_1} < v^s_{\theta_2}$ given that adverse selection is severe. This, in turn, implies that $\underline{p}_{\theta_1} = v^b_{\theta_1}$, as only type-$\theta_1$ sellers find it optimal to visit sub-market $\underline{p}_{\theta_1}$. There are two cases to consider then. Either $\overline{p}_{\theta_1} = \underline{p}_{\theta_1}$ and the equilibrium is separating, or $\overline{p}_{\theta_1} = \underline{p}_{\theta_2} \geq v^s_{\theta_2}$ and the equilibrium is semi-pooling. In the second case Bayes' rule implies that $\underline{p}_{\theta_2} < v^b_{\theta_2}$.

Separating Equilibria

In this case, $supp[\lambda^s_{\theta_1}] = \mathcal{P}_{\theta_1} = \{v^b_{\theta_1}\}$. We claim that $q^s(v^b_{\theta_1}) = 1$. Indeed, given that $U^b = 0$, buyer optimality implies that $q^b(p) = 0$ for all $p < v^b_{\theta_1}$. Frictionless matching then implies that $q^s(p) = 1$ for all $p < v^b_{\theta_1}$. Hence, by seller optimality,

$$q^s(v^b_{\theta_1})(v^b_{\theta_1} - v^s_{\theta_1}) \geq \lim_{p \nearrow v^b_{\theta_1}} q^s(p)(p - v^s_{\theta_1}) = v^b_{\theta_1} - v^s_{\theta_1},$$

which implies the desired result. Now observe $\mathcal{P}_{\theta_2}$ is either empty or a singleton. In the second case, $\mathcal{P}_{\theta_2} = \{v^b_{\theta_2}\}$ and seller optimality implies that

$$0 < q^s(v^b_{\theta_2}) \leq \frac{v^b_{\theta_1} - v^s_{\theta_1}}{v^b_{\theta_1} - v^s_{\theta_1}}.$$

Consequently, the payoff vector U in any separating competitive equilibrium is such that $U^b = 0$, $U^s_{\theta_1} = v^b_{\theta_1} - v^b_{\theta_1}$, and

$$0 \leq U^s_{\theta_2} \leq \left(\frac{v^b_{\theta_1} - v^s_{\theta_1}}{v^b_{\theta_2} - v^s_{\theta_1}}\right)(v^b_{\theta_2} - v^s_{\theta_2}).$$

It is easy to see that for any payoff vector U satisfying the above conditions, there exists a separating competitive equilibrium such that U is the vector of payoffs in this equilibrium.

Semi-Pooling Equilibria

In this case, $supp[\lambda^s_{\theta_1}] = \mathcal{P}_{\theta_1} = \{v^b_{\theta_1}, p\}$ and either $\mathcal{P}_{\theta_2} = \{p\}$ or $\mathcal{P}_{\theta_2} = \{p, v^b_{\theta_2}\}$, where $p \in [v^s_{\theta_2}, v^b_{\theta_2})$. The same argument used in the in the previous case shows that $q^s(v^b_{\theta_1}) = 1$. Thus,

$$q^s(p) = \frac{v^b_{\theta_1} - v^s_{\theta_1}}{p - v^s_{\theta_1}}$$

by seller optimality. This, in turn, implies that if $v^b_{\theta_2} \in \mathcal{P}_{\theta_2}$, then

$$q^s(v^b_{\theta_2}) = \frac{p - v^s_{\theta_2}}{v^b_{\theta_2} - v^s_{\theta_2}} \cdot \frac{v^b_{\theta_1} - v^s_{\theta_1}}{p - v^s_{\theta_1}},$$

where we used seller optimality one more time. Since $p \mapsto (p - v^s_{\theta_2})/(p - v^s_{\theta_1})$ is strictly increasing in p, we then have that the payoff vector U in any semi-pooling competitive equilibrium is such that $U^b = 0$, $U^s_{\theta_1} = v^b_{\theta_1} - v^b_{\theta_1}$, and

$$0 \leq U^s_{\theta_2} < \left(\frac{v^b_{\theta_1} - v^s_{\theta_1}}{v^b_{\theta_2} - v^s_{\theta_1}}\right)(v^b_{\theta_2} - v^s_{\theta_2}).$$

Consequently, for any semi-pooling competitive equilibrium there exists a separating competitive equilibrium that generates the same payoffs for buyers and sellers, and the most efficient separating competitive equilibrium generates higher welfare than any semi-pooling competitive equilibrium.

5 Constrained Efficiency

In this section we discuss constrained efficiency. We first define the second-best and establish some of its properties. We then discuss the constrained efficiency of competitive equilibria. We show that competitive equilibria are generically constrained inefficient when adverse selection is severe and that separating competitive equilibria are always constrained inefficient. We conclude by showing that one can achieve the second-best using budget-balanced taxes and transfers to sellers.

5.1 Second-Best

We consider anonymous mechanisms. A direct mechanism for the sellers is a pair (π, p) with $\pi : \Theta \to [0, 1]$ and $p : \Theta \to \Delta(\mathbb{R}_+)$ such that $\pi(\theta)$ and $p(\theta)$ are, respectively, the probability of trade and the (random) payment in case of trade for a seller who announces that his type is θ. Fix a direct mechanism (π, p) and for each $i \in \{1, \ldots, N\}$ let $\pi_i = \pi(\theta_i)$ and $p_i = \mathbb{E}[p|\theta = \theta_i]$. The mechanism is incentive compatible and individually rational if

$$\pi_i(p_i - v^s_{\theta_i}) \geq \max\left\{0, \max_{j \neq i} \pi_j(p_j - v^s_{\theta_i})\right\}$$

for all $i \in \{1, \ldots, N\}$. The mechanism is budget balanced if

$$\sum_{i=1}^{N} f_{\theta_i} \pi_i (v^b_{\theta_i} - p_i) \geq 0.$$

The last constraint ensures that buyers are (ex-ante) willing to pay for the transfers to the sellers that are implied by the mechanism. The welfare associated with the mechanism is

$$\sum_{i=1}^{N} f_{\theta_i}\pi_i(v^b_{\theta_i} - v^s_{\theta_i}).$$

A standard argument shows that we can restrict attention to individually rational and incentive compatible direct mechanisms. Anonymity and the continuum population assumption imply that we can further restrict attention to direct mechanisms in a which a seller's outcome depends only on his type announcement. Indeed, when the mechanism is anonymous, a seller's message does not affect the aggregate outcome when there is a continuum of other sellers. Quasi-linearity of preferences implies that we can also restrict attention to direct mechanisms (π, p) in which the prices $p(\theta)$ are deterministic, and so $p_i = p(\theta_i)$ for all $i \in \{1, \ldots, N\}$. In a slight abuse of terminology, we refer to such mechanisms as direct deterministic mechanisms. The planner's problem is to find an individually rational, incentive compatible, and budget balanced direct deterministic mechanism that maximizes welfare.

We define the second-best differently from Gale (1996). In the Appendix we define the second-best following Gale's approach and show that both definitions are equivalent.

5.2 Characterizing the Second-Best

We now establish some properties of the second-best. Since a direct deterministic mechanism with $p_i > v^s_{\theta_i}$ for some $i \in \{1, \ldots, N\}$ is individually rational only if $\pi_i = 0$, it follows that the planner's problem always has a solution. It is immediate to see that the first-best is implementable if, and only if, adverse selection is not severe.[12] The next result shows that at least the two lowest types of good transact at the second-best when adverse selection is severe and that in this case the highest type of good that trades at the second-best trades at its reservation value for the seller.

Proposition 9. *Suppose that adverse selection is severe and let (π, p) be a solution to the planner's problem. Then $\pi_1, \pi_2 > 0$. Moreover, if $j = \max\{i \in \{1, \ldots, N\} : \pi_i > 0\}$, then $p_j = v^s_{\theta_j}$.*

[12]Indeed, if $\overline{v}^b \geq v^s_{\theta_N}$, then the direct mechanism (π, p) such that $\pi_i \equiv 1$ and $p_i \equiv v^s_{\theta_N}$ is individually rational, incentive compatible, and budget balanced. Moreover, any mechanism with $\pi_i \equiv 1$ is incentive compatible if, and only if, there exists $p \in \mathbb{R}_+$ such that $p_i \equiv p$. Since in this case $p \geq v^s_{\theta_N}$ is a necessary condition for individual rationality, we then have that $\sum_{i=1}^N f_{\theta_i}\pi_i(v^b_{\theta_i} - p_i) \leq \sum_{i=1}^N f_{\theta_i}(v^b_{\theta_i} - v^s_{\theta_N}) < 0$ when adverse selection is severe.

Suppose that adverse selection is severe. The following direct deterministic mechanism is individually rational, incentive compatible, and budget balanced: $p_i = v^b_{\theta_i}$ for all $i \in \{1, \ldots, N\}$, $\pi_1 = 1$, and $\pi_{i+1}(v^b_{\theta_{i+1}} - v^s_{\theta_i}) = \pi_i(v^b_{\theta_i} - v^s_{\theta_i})$ for all $i \in \{1, \ldots, N-1\}$. This mechanism generates the same allocation as the separating competitive equilibrium of Proposition 1. Now notice that among the individually rational, incentive compatible, and budget balanced direct deterministic mechanisms with the property that $\pi_1 \geq 0$ and $\pi_i = 0$ for all $i \geq 2$, the ones that maximize welfare have $\pi_1 = 1$. Thus, any direct deterministic mechanism with $\pi_i = 0$ for all $i \geq 2$ is suboptimal.

Consider now an individually rational, incentive compatible, and budget balanced direct deterministic mechanism with $\pi_i > 0$ for some $i \in \{2, \ldots, N\}$. Standard arguments show that incentive compatibility implies that the probabilities of trade π_i are nonincreasing in i and the prices p_i are nondecreasing in i. Then there exists $j \in \{1, \ldots, N\}$ such that $\pi_i > 0$ for all $i \leq j$ and $\pi_i = 0$ for all $i > j$. Individual rationality implies that $p_j \geq v^s_{\theta_j}$. We claim that $p_j > v^s_{\theta_j}$ is suboptimal. Without loss, we can assume that either the local upward incentive compatibility constraint

$$\pi_{j-1}(p_{j-1} - v^s_{\theta_{j-1}}) \geq \pi_j(p_j - v^s_{\theta_{j-1}}) \tag{7}$$

holds with strict inequality or the local downward incentive compatibility constraint

$$\pi_j(p_j - v^s_{\theta_j}) \geq \pi_{j-1}(p_{j-1} - v^s_{\theta_j}). \tag{8}$$

holds with strict inequality. When both these constraints bind, $\pi_{j-1} = \pi_j$ and $p_{j-1} = p_j$, and we can treat type-θ_{j-1} sellers and type-θ_j sellers as a single type.

Suppose first that (7) holds with strict inequality and (8) holds with equality. Summing (7) with (8), we obtain that $\pi_{j-1} > \pi_j$, and so $p_{j-1} < p_j$. Consider the following change in the mechanism: increase π_j and decrease p_j so that $\pi_j(p_j - v^s_{\theta_j})$ remains constant. Given that

$$\pi_j(p_j - v^s_{\theta_{j-1}}) = \pi_j(p_j - v^s_{\theta_j}) + \pi_j(v^s_{\theta_j} - v^s_{\theta_{j-1}}),$$

this change increases $\pi_j(p_j - v^s_{\theta_{j-1}})$, and so does not violate (7) if it is small enough. Since

$$\frac{d\pi_j}{dp_j}(p_j - v^s_{\theta_j}) + \pi_j = 0$$

by assumption, and so $d\pi_j/dp_j < 0$, we also have that

$$\frac{d}{dp_j}\pi_j(v^b_{\theta_j} - p_j) = \frac{d\pi_j}{dp_j}(v^b_{\theta_j} - p_j) - \pi_j = \frac{d\pi_j}{dp_j}(v^b_{\theta_j} - v^b_{\theta_j}) < 0.$$

Thus, this change does not violate budget balance as well. Hence, the change in question is feasible and increases welfare. Suppose now that (8) holds with strict inequality. In this case, reducing p_j is feasible and increases welfare. This shows that $p_j > v^s_{\theta_j}$ is suboptimal.

The result that when adverse selection is severe the highest type of good that trades at the second-best trades at the lowest price possible plays a key role in our discussion of the constrained inefficiency of competitive equilibria. In the remainder of this part we discuss conditions under which all types of good trade at the second-best when adverse selection is severe.

Let $N \geq 3$ and suppose that: (*i*) gains from trade are nondecreasing in θ; and (*ii*) seller valuations are (weakly) concave in θ, that is, the difference $v^s_{\theta_i} - v^s_{\theta_{i-1}}$ is nonincreasing in i. Consider an individually rational, incentive compatible, and budget balanced direct deterministic mechanism with $\pi_i > 0$ for some $i \in \{2, \ldots, N-1\}$ and $\pi_j = 0$ for all $j > i$. We know from above that $p_i > v^s_{\theta_i}$ is suboptimal, so assume that $p_i = v^s_{\theta_i}$.

Now consider the following change in the mechanism: $(\pi_i, p_i) \mapsto (\pi_i - \delta, p_i + \varepsilon)$ such that

$$(\pi_i - \delta)(v^s_{\theta_i} - v^s_{\theta_{i-1}} + \varepsilon) = \pi_i(v^s_{\theta_i} - v^s_{\theta_{i-1}}) \Leftrightarrow (\pi_i - \delta)\varepsilon = \delta(v^s_{\theta_i} - v^s_{\theta_{i-1}})$$

and $(\pi_{i+1}, p_{i+1}) \mapsto (\pi_{i+1}, v^s_{\theta_{i+1}})$ such that $\pi_{i+1}(v^s_{\theta_{i+1}} - v^s_{\theta_i}) = (\pi_i - \delta)\varepsilon$. This change respects budget balance. It also respects individual rationality, as it increases the payoff of the type-θ_i sellers from zero to $(\pi_i - \delta)\varepsilon$ and maintains the payoff of the type-θ_{i+1} sellers equal to zero. Moreover, this change maintains the local upward incentive compatibility constraint of the type-θ_{i-1} sellers, respects both the local upward incentive compatibility constraint of the type-θ_i sellers and the local downward incentive compatibility constraint of the type-θ_{i+1} sellers (as long as $v^s_{\theta_i} + \varepsilon \leq v^s_{\theta_{i+1}}$), and relaxes the local downward incentive compatibility constraint of the type-θ_i sellers. Thus, the change is feasible. To conclude, observe that the welfare change associated with this change is

$$-\delta \underbrace{(v^b_{\theta_i} - v^s_{\theta_i})}_{\Delta_i} + \pi_{i+1} \underbrace{(v^b_{\theta_{i+1}} - v^s_{\theta_{i+1}})}_{\Delta_{i+1}} = \delta \left[\frac{v^s_{\theta_i} - v^s_{\theta_{i-1}}}{v^s_{\theta_{i+1}} - v^s_{\theta_i}} \Delta_{i+1} - \Delta_i \right], \tag{9}$$

which is positive by assumption.

5.3 Constrained Inefficiency of Competitive Equilibria

We now discuss constrained efficiency of competitive equilibria. We know that if adverse selection is not severe, then there exists a pooling competitive equilibrium that achieves the first-best. Moreover, since all non-pooling competitive equilibria involve rationing, such equilibria are constrained inefficient when adverse selection is not severe.

Suppose that adverse selection is severe. We know from Lemma 1 and Proposition 3 that as far as welfare is concerned, we can restrict attention to pure competitive equilibria in which $U^b = 0$ and only a finite number of sub-markets open in equilibrium. Consider such an equilibrium and let $\mathcal{P} = \{p_1, \ldots, p_M\}$, with $M \geq 1$, be the set of sub-markets that open in equilibrium. By Proposition 2, there exists a partition $(\Theta_1, \ldots, \Theta_{M+1})$ of Θ such that: (*i*) $\theta < \theta'$ for all $\theta \in \Theta_i$ and $\theta' \in \Theta_j$ if $i < j$; (*ii*) $\lambda^s_\theta(\{p_i\}) = 1$ if, and only if, $\theta \in \Theta_i$; and (*iii*) $\lambda^s_\theta(\mathcal{P}) = 0$ for all $\theta \in \Theta_{M+1}$.[13] Since $U^b = 0$ and $q^b(p) > 0$ for all $p \in \mathcal{P}$, it then follows that

$$p_j = \frac{\sum_{\theta \in \Theta_j} f_\theta v^b_\theta}{\sum_{\theta \in \Theta_j} f_\theta}$$

for all $j \in \{1, \ldots, M\}$. In particular, if $\theta_M = \max\{\theta : \theta \in \Theta_M\}$ is the highest type of seller that trades in equilibrium, then $p_M = v^s_{\theta_M}$ if, and only if,

$$\sum_{\theta \in \Theta_M} f_\theta v^b_\theta = \left(\sum_{\theta \in \Theta_M} f_\theta \right) v^s_{\theta_M}. \tag{10}$$

By Proposition 9, condition (10) is a necessary condition for the equilibrium under consideration to implement the second-best. Notice that if the equilibrium is separating, then (10) does not hold, as it would reduce to $v^b_{\theta_M} = v^s_{\theta_M}$.

For any set $\Theta = \{\theta_1, \ldots, \theta_N\}$ of seller types, payoff vectors $v^b = (v^b_{\theta_1}, \ldots, v^b_{\theta_N})$ for buyers, and payoff vectors $v^s = (v^s_{\theta_1}, \ldots, v^s_{\theta_N})$ for sellers, let

$$\mathcal{F}_{AS} = \left\{ f = (f_\theta)_{\theta \in \Theta} : \sum_{\theta \in \Theta} f_\theta v^b_\theta < v^b_{\theta_N} \right\}$$

be the set of probability distributions over Θ for which adverse selection is severe. Now let $\mathcal{F}^*$ be the subset of $\mathcal{F}_{AS}$ such that $f \in \mathcal{F}^*$ if, and only if, there exists $M \in \{1, \ldots, N\}$ and $j \leq M$ such

[13] Since all competitive equilibria are non-pooling when adverse selection is severe, Θ_2 is non-empty when $M = 1$.

that (10) holds when $\Theta_M = \{j, \dots, M\}$. It follows from the above discussion that if $f \notin \mathcal{F}^*$, then all competitive equilibria do not implement the second-best. Since $\mathcal{F}^*$ is a non-generic subset of $\mathcal{F}_{AS}$, we have the following result.

Proposition 10. *For almost all $f \in \mathcal{F}_{AS}$, all competitive equilibria are constrained inefficient. Moreover, separating competitive equilibria are always constrained inefficient.*

5.4 Competitive Equilibria with Taxes

We now define competitive equilibria with taxes and show that for every individually rational, incentive compatible, and budget balanced direct deterministic mechanism there exists a competitive equilibrium with taxes that implements the same allocation as the mechanism. This equilibrium is such that only sellers are taxed.

Suppose that in every submarket $p \in \mathbb{R}_+$, the planner can impose a tax $\tau^b = \tau^b(p)$ on buyers and a tax $\tau^s = \tau^s(p)$ on sellers. In this way, the prices to buyers and sellers in sub-market $p \in \mathbb{R}_+$ are, respectively, $p^b = p + \tau^b$ and $p^s = p - \tau^s$. A *tax schedule* is a pair $\tau = (\tau^b, \tau^s)$ of functions from $\mathbb{R}^*$ into $\mathbb{R}$ such that $\tau^b(p)$ and $\tau^s(p)$ are, respectively, the tax that buyers and sellers pay when they trade in sub-market p, with the convention that $\tau^b(n) = \tau^s(n) = 0$. Negative taxes are possible. Trading rules, trading outcomes, beliefs, and payoff vectors are the same as before.

Definition. *A list $(\lambda, q, \tau, \beta, U)$, where λ is a trading rule, q is a trading outcome, τ is a tax schedule, β is a belief, and U is a vector of payoffs is a competitive equilibrium with taxes if it satisfies the following conditions.*

1. Buyer optimality: *For each $p \in \mathbb{R}^*$,*

$$U^b \geq q^b(p)\left(\sum_\theta \beta_\theta(p)v_\theta^b - p - \tau^b(p)\right), \text{ with equality holding } \lambda^b\text{-almost surely.}$$

2. Seller optimality: *For each $\theta \in \Theta$ and $p \in \mathbb{R}^*$,*

$$U_\theta^s \geq q^s(p)(p - \tau^s(p) - v_\theta^s), \text{ with equality holding } \lambda_\theta^s\text{-almost surely.}$$

3. Market clearing: *For each $\theta \in \Theta$ and $P \subseteq \mathbb{R}_+$,*

$$\int_P \beta_\theta(p)d\mu^b(p) = f_\theta \mu_\theta^s(P).$$

4. Monotonic Trading: *For every* $p, p' \in \mathbb{R}_+$, $q^s(p') \leq q^s(p)$ *implies that* $q^b(p') \geq q^b(p)$.

5. Budget Balance:

$$\int_{\mathbb{R}_+} \tau^s(p)d\overline{\mu}^s(p) + \int_{\mathbb{R}_+} \tau^b(p)d\mu^b(p) \geq 0$$

The first four conditions are the same as before, except that now buyers and sellers pay taxes when they transact. In order to understand the last condition, notice that

$$\int_{\mathbb{R}_+} \tau^s(p)d\overline{\mu}^s(p)$$

is the net revenue from taxing the sellers and

$$\int_{\mathbb{R}_+} \tau^b(p)d\mu^b(p)$$

is the net revenue from taxing the buyers. Budget balance requires that the total net revenue from taxing buyers and sellers be nonnegative.

The next result shows that for every individually rational, incentive compatible and budget balanced direct deterministic mechanism there exists a competitive equilibrium with taxes that implements the same allocation as the mechanism.

Proposition 11. *For any individually rational, incentive compatible, and budget balanced direct deterministic mechanism there exists a competitive equilibrium with taxes that implements the same allocation as the mechanism.*

The proof is constructive. Consider an individually rational, incentive compatible, and budget balanced direct deterministic mechanism (π, p). Incentive compatibility implies that there exist a set $\mathcal{P} = \{p_1, \ldots, p_M\}$ of prices and a partition $\Pi = (\Theta_1, \ldots, \Theta_M)$ of Θ such that: (*i*) $p_1 < \cdots < p_M$; (*ii*) $\theta < \theta'$ for all $\theta \in \Theta_i$ and $\theta' \in \Theta_j$ if $j > i$; and (*iii*) $p(\theta) = p_j$ if $\theta \in \Theta_j$. Moreover, if θ and θ' belong to the same element of Π, then $\pi(\theta) = \pi(\theta')$. Individual rationality implies that for each $j \in \{1, \ldots, M\}$, $p_j \geq v^s_\theta$ for all $\theta \in \Theta_j$. Budget balancedness implies that

$$\sum_{j=1}^{M} \sum_{\theta \in \Theta_j} f_\theta(v^b_\theta - p_j) \geq 0.$$

We define a competitive equilibrium with taxes as follows. For each $j \in \{1, \ldots, M\}$, let

$$\hat{p}_j = \frac{\sum_{\theta \in \Theta_j} f_\theta v_\theta^b}{\sum_{\theta \in \Theta_j} f_\theta}.$$

Notice that $\hat{p}_1 < \cdots < \hat{p}_M$. The trading rule for sellers is such that $\lambda_\theta^s(\{\hat{p}_j\}) = 1$ if $\theta \in \Theta_j$. The trading rule for buyers is such that $\lambda^b(\{\hat{p}_j\}) = (\sum_{\theta \in \Theta_j} f_\theta)\pi(p_j)$ and $\lambda^b(\{n\}) = 1 - \sum_{j=1}^N \lambda^b(\{\hat{p}_j\})$. The trading outcome for sellers is such that: (*i*) $q^s(p) = \pi(p_1)$ if $p \le \hat{p}_1$; (*ii*) $q^s(p) = \pi(p_j)$ if $p \in (\hat{p}_{j-1}, \hat{p}_j]$ for $j \in \{2, \ldots, M\}$; and (*iii*) $q^s(p) = 0$ if $p > \hat{p}_M$. The trading outcome for buyers is such that $q^b(p) = 1$ if $p \ge \hat{p}_1$ and $q^b(p) = 0$ otherwise. By construction, the set of sub-markets that open in equilibrium is $\{\hat{p}_1, \ldots, \hat{p}_M\}$ and trading is monotonic (but not necessarily frictionless). The tax schedule for sellers is $\tau^s(p) = \hat{p}_j - p_j$ if $p = \hat{p}_j$ and $\tau^s(p) = v_{\theta_N}^b$ otherwise. The tax schedule for buyers is $\tau^b(p) \equiv 0$. The belief is such that: (*i*) $\beta_\theta(\hat{p}_j) = f_\theta / (\sum_{\theta \in \Theta_j} f_\theta)$ if $\theta \in \Theta_j$; (*ii*) $\beta_\theta(p) = \beta_\theta(\hat{p}_1)$ if $p < \hat{p}_1$; (*iii*) $\beta_\theta(p) = \beta_\theta(\hat{p}_j)$ if $p \in (\hat{p}_j, \hat{p}_{j+1})$ and $j \in \{1, \ldots, M-1\}$; and (*iv*) $\beta_\theta(p) = \beta_\theta(\hat{p}_M)$ if $p > \hat{p}_M$. To finish the description of the equilibrium, the vector of payoffs is such that $U^b = 0$ and $U_\theta^s = \pi(p_j)(p_j - v_\theta^s)$ if $\theta \in \Theta_j$. By construction, $(\lambda, q, \tau, \beta, U)$ implements the same allocation as the mechanism (π, p).

We claim that $(\lambda, q, \tau, \beta, U)$ defined above is a competitive equilibrium with taxes. By construction, the buyers' payoff is zero in any sub-market that opens in equilibrium and is nonpositive in any sub-market that does not open in equilibrium. Thus, buyer optimality holds. On the other hand, the payoff to a type-θ seller who goes to sub-market $\hat{p}_j$ is $\pi(p_j)(p_j - v_\theta^s)$ and is nonpositive if he goes to any other sub-market. Thus, since the mechanism (π, p) is incentive compatible, seller optimality also holds. Now observe that for all $j \in \{1, \ldots, M\}$,

$$\sum_{\theta \in \Theta_j} f_\theta q^s(\hat{p}_j) \lambda_\theta^s(\{\hat{p}_j\}) = (\sum_{\theta \in \Theta_j} f_\theta) \pi(p_j) = q^b(\hat{p}_j) \lambda^b(\{\hat{p}_j\}).$$

Thus, market clearing holds as well. To conclude notice that

$$\int_{\mathbb{R}_+} \tau^s(p) d\overline{\mu}^s(p) + \int_{\mathbb{R}_+} \tau^b(p) d\mu^b(p)$$
$$= \sum_{j=1}^M \sum_{\theta \in \Theta_j} (\hat{p}_j - p_j) = \sum_{j=1}^M \sum_{\theta \in \Theta_j} f_\theta (v_\theta^b - p_j) \ge 0,$$

where the inequality follows from the fact that the mechanism (π, p) is budget balanced. Thus, $(\lambda, q, \tau, \beta, U)$ is indeed a competitive equilibrium with taxes.

The next result is an immediate corollary of Proposition 11.

Corollary 3. *There exists a competitive equilibrium with taxes that implements the second-best.*

References

[1] Gale, D. (1996): “Equilibria and Pareto Optima for Markets with Adverse Selection," *Economic Theory*, 7, 207-235.

[2] Guerrieri, V., Shimer, R., and R. Wright (2010): “Adverse Selection in Competitive Search Equilibrium," *Econometrica*, 78(6), 1823-1862.

Appendix

Section 2

Proof of Equation (5)

Buyer and seller optimality imply that

$$\begin{aligned} U^b + \sum_{\theta \in \Theta} f_\theta U_\theta^s &= \int_{\mathbb{R}^*} q^b(p)(v_\theta^b - p) d\lambda^b(p) + \sum_{\theta \in \Theta} f_\theta \int_{\mathbb{R}^*} q^s(p)(p - v_\theta^s) d\lambda_\theta^s(p) \\ &= \int_{\mathbb{R}^*} (v_\theta^b - p) d\mu^b(p) + \sum_{\theta \in \Theta} f_\theta \int_{\mathbb{R}^*} (p - v_\theta^s) d\mu_\theta^s(p). \end{aligned}$$

Since $\mu^b = \sum_{\theta \in \Theta} f_\theta \mu_\theta^s$ by market clearing, we then have that

$$U^b + \sum_{\theta \in \Theta} f_\theta U_\theta^s = \sum_{\theta \in \Theta} f_\theta \int_{\mathbb{R}^*} (v_\theta^b - v_\theta^s) d\mu_\theta^s(p),$$

which implies the desired result.

Relation to Guerrieri, Shimer, and Wright (2010)

We first show that

$$\frac{q^s(p)}{q^b(p)} = \lim_{\varepsilon \to 0} \frac{\lambda^b([p-\varepsilon, p+\varepsilon])}{\overline{\lambda}^s([p-\varepsilon, p+\varepsilon])}$$

for $\overline{\lambda}^s$-almost all $p \in \mathbb{R}_+$. In an abuse of notation, let μ^b, $\overline{\mu}^s$, λ^b, and $\overline{\lambda}^s$ denote the restrictions of these same measures to $\mathbb{R}_+$. By construction, $\mu^b \ll \lambda^b$, that is, μ^b is absolutely continuous with respect to λ^b, and $q^b = d\mu^b / d\lambda^b$, the Radon-Nikodym derivative of μ^b with respect to λ^b. Likewise, $\overline{\mu}^s \ll \overline{\lambda}^s$ and $q^s = d\overline{\mu}^s / d\overline{\lambda}^s$.[14] Now observe that we can assume that $\lambda^b \ll \mu^b$, so that

$$\frac{d\lambda^b}{d\mu^b} = \left(\frac{d\mu^b}{d\lambda^b} \right)^{-1} = \frac{1}{q^b}.$$

Indeed, if there exits $P \subseteq \mathbb{R}_+$ such that $\mu^b(P) = 0$ and $\lambda^b(P) > 0$, then $q^b(p) = 0$ for λ^b-almost all $p \in P$. This, in turn, implies that $U^b = 0$ by buyer optimality, in which case we can set $\lambda^b(P)$ to zero and increase $\lambda^b(\{n\})$ by $\lambda^b(P)$, as the decision of a buyer to go to a sub-market in which

[14] The measures μ^b, $\overline{\mu}^s$, λ^b, and $\overline{\lambda}^s$ are σ-finite, and so the Radon-Nikodym derivatives in question are well-defined.

the probability that he trades is zero is equivalent to the decision of going to sub-market n. Since $\mu^b = \overline{\mu}^s$ by market clearing, we then have that $\lambda^b \ll \mu^b = \overline{\mu}^s \ll \overline{\lambda}^s$, and so

$$\frac{d\lambda^b}{d\overline{\lambda}^s} = \frac{d\lambda^b}{d\mu^b} \cdot \frac{d\overline{\mu}^s}{d\overline{\lambda}^s} = \frac{q^s}{q^b}.$$

The desired result now follows from the Lebesgue-Besicovitch differentiation theorem.

We now show that monotonic trading implies that there exist matching functions m^s for the sellers and m^b for the buyers such that $m^s(\Gamma(p)) = q^s(p)$ and $m^b(\Gamma(p)) = q^b(p)$ for all $p \in \mathbb{R}_+$. Let $\mathcal{I}$ be the image of q^s. We claim that there exists a nonincreasing function $f : \mathcal{I} \to [0,1]$ such that $f(q^s(p)) = q^b(p)$ for all $p \in \mathbb{R}_+$. For each $y \in \mathcal{I}$, let $\Omega(y) = \{p : q^s(p) = y\}$. Now define $f : \mathcal{I} \to [0,1]$ to be such that $f(y)$ is the value that q^b assumes in $\Omega(y)$. The function f is well-defined since monotonic trading implies that q^b is constant in $\Omega(y)$ for all $y \in \mathcal{I}$. By construction, $f(q^s(p)) = q^b(p)$. Now let $y, y' \in \mathcal{I}$ be such that $y > y'$ and $p, p' \in \mathbb{R}_+$ be such that $p \in \Omega(y)$ and $p' \in \Omega(y')$. Since $q^s(p') < q^s(p)$, monotonic trading implies that $f(y') = q^b(p') \geq q^b(p) = f(y)$, and so f is nonincreasing.

It follows from the previous paragraph that $\Gamma(p) = h(q^s(p))$ for all $p \in \mathbb{R}_+$, where $h : \mathcal{I} \to \mathbb{R}_+ \cup \{\infty\}$ is such that $h(y) = y/f(y)$. The function h is invertible given that f is nonincreasing. Now define $m^* : h(\mathcal{I}) \to [0,1]$ to be such that $m^*(\gamma) = h^{-1}(\gamma)$. Notice that $m^*(\Gamma(p)) = q^s(p)$ and $m^*(\Gamma(p))/\Gamma(p) = q^b(p)$. By construction, m^* is strictly increasing in $h(\mathcal{I})$. Moreover, since

$$\frac{m^*(\gamma)}{\gamma} = \frac{m^*(\gamma)}{h(m^*(\gamma))} = f(m^*(\gamma)),$$

the function $\gamma \mapsto m^*(\gamma)/\gamma$ is nonincreasing in $h(\mathcal{I})$. To conclude, let $m^s : \mathbb{R}_+ \cup \{\infty\} \to [0,1]$ be a monotonic extension of m^* to $\mathbb{R}_+ \cup \{\infty\}$ and $m^b : \mathbb{R}_+ \cup \{\infty\} \to [0,1]$ by a monotonic extension of $\gamma \mapsto m^*(\gamma)/\gamma$ to $\mathbb{R}_+ \cup \{\infty\}$.[15] The functions m^s and m^b are the desired matching functions.

Section 3

Proof of Proposition 1

We establish seller optimality for the separating equilibrium under consideration. We first claim that for all $\theta \in \Theta$, a type-θ seller has no incentive to deviate to $p > v^b_\theta$. Indeed, since $U^s_\theta > 0$ for

[15] For instance, we can take m^s to be such that $m(\gamma) = \inf\{m^*(\gamma') : \gamma \in h(\mathcal{I}) \text{ and } \gamma > \gamma'\}$, with the convention that $m(\gamma) = 1$ if $\gamma > \gamma'$ for all $\gamma' \in h(\mathcal{I})$. The function m^b can be defined similarly.

all $\theta \in \Theta$, no seller has an incentive to direct his trade to $p > v^b_{\theta_N}$. Fix $i \in \{1, \dots, N-1\}$ and suppose, by induction, that there exists $j \in \{i+1, \dots, N\}$ such that $q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_i}) \le U^s_{\theta_i}$; that is, a type-$\theta_i$ seller has no incentive to direct his trade to $p \in (v^b_{\theta_{j-1}}, v^b_{\theta_j}]$. The induction hypothesis holds for $j = i+1$. Moreover, if $j+1 \le N$, then

$$\begin{aligned}
q^s(v^b_{\theta_{j+1}})(v^b_{\theta_{j+1}} - v^s_{\theta_i}) &= q^s(v^b_{\theta_{j+1}})(v^b_{\theta_{j+1}} - v^s_{\theta_j}) + q^s(v^b_{\theta_{j+1}})(v^s_{\theta_j} - v^s_{\theta_i}) \\
&= q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_j}) + q^s(v^b_{\theta_{j+1}})(v^s_{\theta_j} - v^s_{\theta_i}) \\
&< q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_j}) + q^s(v^b_{\theta_j})(v^s_{\theta_j} - v^s_{\theta_i}) \\
&= q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_i}),
\end{aligned}$$

where the second equality follows from the definition of $q^s(v^b_{\theta_{j+1}})$ and the inequality follows from $q^s(v^b_{\theta_{j+1}}) < q^s(v^b_{\theta_j})$. So, the induction hypothesis holds for $j+1$ if it holds for j. This establishes the desired result.

We now claim that for all $\theta \in \Theta$, a type-θ seller has no incentive to deviate to $p < v^b_\theta$. Clearly, a type-θ_1 seller has no incentive to direct his trade to $p < v^b_{\theta_1}$. Fix $i \in \{2, \dots, N\}$. It is sufficient to show that a type-θ_i seller has no incentive to direct his trade to $p \in \{v^b_{\theta_1}, \dots, v^b_{\theta_{i-1}}\}$. Suppose, by induction, that there exists $j \in \{1, \dots, i-1\}$ such that $q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_i}) \le U^s_{\theta_i}$; that is, a type-$\theta_i$ seller has no incentive to direct his trade to $p = v^b_{\theta_j}$. Since

$$\begin{aligned}
q^s(v^b_{\theta_{i-1}})(v^b_{\theta_{i-1}} - v^s_{\theta_i}) &= q^s(v^b_{\theta_{i-1}})(v^b_{\theta_{i-1}} - v^s_{\theta_{i-1}}) + q^s(v^b_{\theta_{i-1}})(v^s_{\theta_{i-1}} - v^s_{\theta_i}) \\
&= q^s(v^b_{\theta_i})(v^b_{\theta_i} - v^s_{\theta_{i-1}}) + q^s(v^b_{\theta_{i-1}})(v^s_{\theta_{i-1}} - v^s_{\theta_i}) \\
&< q^s(v^b_{\theta_i})(v^b_{\theta_i} - v^s_{\theta_i}),
\end{aligned}$$

where the inequality follows from $q^s(v^b_{\theta_{i-1}}) > q^s(v^b_{\theta_i})$, the induction hypothesis holds for $j = i-1$. Moreover, if $j-1 \ge 1$, then

$$\begin{aligned}
q^s(v^b_{\theta_{j-1}})(v^b_{\theta_{j-1}} - v^s_{\theta_i}) &= q^s(v^b_{\theta_{j-1}})(v^b_{\theta_{j-1}} - v^s_{\theta_j}) + q^s(v^b_{\theta_{j-1}})(v^s_{\theta_j} - v^s_{\theta_i}) \\
&= q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_j}) + q^s(v^b_{\theta_{j-1}})(v^s_{\theta_j} - v^s_{\theta_i}) \\
&< q^s(v^b_{\theta_j})(v^b_{\theta_j} - v^s_{\theta_i}).
\end{aligned}$$

Hence, the induction hypothesis holds for $j-1$ if it holds for j. This concludes the proof.

Proof of Proposition 2

Suppose, by contradiction, that there exist $i > j$ and $p' > p$ with $p' \in supp[\lambda^s_{\theta_j}]$ and $p \in supp[\lambda^s_{\theta_i}]$. Now let $0 < \varepsilon < (p' - p)/2$. By seller optimality, there exist $\hat{p} \in (p - \varepsilon, p + \varepsilon)$ and $\tilde{p} \in (p' - \varepsilon, p' + \varepsilon)$ such that:

$$U^s_{\theta_i} = q^s(\hat{p})(\hat{p} - v^s_{\theta_i}) \geq q^s(\tilde{p})(\tilde{p} - v^s_{\theta_i});$$
$$U^s_{\theta_j} = q^s(\tilde{p})(\tilde{p} - v^s_{\theta_j}) \geq q^s(\hat{p})(\hat{p} - v^s_{\theta_j}).$$

Summing the two inequalities, we obtain that

$$(q^s(\hat{p}) - q^s(\tilde{p}))(v^s_{\theta_j} - v^s_{\theta_i}) \geq 0.$$

This, in turn, implies that $q^s(\hat{p}) \leq q^s(\tilde{p})$. Given that $\tilde{p} > \hat{p}$, the type-θ_i sellers can then profitably deviate by going to the sub-market $\tilde{p}$, a contradiction. This concludes the proof.

Proof of Corollary 1

Seller optimality implies that $q^s(p) < q^s(p')$ for all $p, p' \in supp[\overline{\mu}^s]$ such that $p > p'$. Now observe, by Proposition 2, that if $\theta > \theta'$, then λ^s_θ dominates $\lambda^s_{\theta'}$ in the strict first-order stochastic sense as long as these measures do not coincide. Hence, $\theta > \theta'$ implies

$$\int_{\mathbb{R}_+} q^s(p) d\lambda^s_{\theta'} \geq \int_{\mathbb{R}_+} q^s(p) d\lambda^s_\theta,$$

with strict inequality if λ^s_θ and $\lambda^s_{\theta'}$ do not coincide. This implies the desired result.

Section 5

Alternative Definition of the Second-Best

Gale (1996) defines the second-best in a different way. According to him, a (direct) mechanism is a pair (μ, λ) of stochastic kernels from Θ into $\mathbb{R}^*$ such that for each $\theta \in \Theta$ and each $P \subseteq \mathbb{R}^*$, $\mu(\theta, P)$ is the probability that a seller who announces that his type is θ trades in a sub-market in P and $\lambda(\theta, P)$ is the probability that a buyer trades with a type-θ seller in a sub-market in P. Given a mechanism (μ, λ), the (expected) payoff to a type-θ seller who announces that his type is θ' is

$$U^s(\theta, \theta') = \int_{\mathbb{R}_+} (p - v^s_\theta) \mu(\theta', dp),$$

while the buyers' payoff is

$$U^b = \sum_{\theta \in \Theta} \int_{\mathbb{R}_+} (v^b_\theta - p)\lambda(\theta, dp).$$

The welfare associated with the mechanism (μ, λ) is

$$\sum_{\theta \in \Theta} \int_{\mathbb{R}_+} (v^b_\theta - v^s_\theta)\lambda(\theta, dp).$$

In order to distinguish the type of mechanism that Gale considers from the type of mechanism that we consider, we refer to the mechanisms that Gale considers are matching mechanisms.

A matching mechanism (μ, λ) is incentive compatible if $U^s(\theta, \theta) \geq U^s(\theta, \theta')$ for all $(\theta, \theta') \in \Theta^2$, is individually rational if $U^b \geq 0$ and $U^s(\theta, \theta) \geq 0$ for all $\theta \in \Theta$, and satisfies market clearing if $f_\theta \mu(\theta, P) = \lambda(\theta, P)$ for all $\theta \in \Theta$ and $P \subseteq \mathbb{R}_+$. The planner's problem is to find an incentive compatible and individually rational matching mechanism (μ, λ) satisfying market clearing that maximizes welfare. We now show that this alternative definition of second-best coincides with the definition of second-best that we use in the main text.

Fix a matching mechanism (μ, λ) and define the deterministic mechanism (π, p) to be such that $\pi(\theta) = \mu(\theta, \mathbb{R}_+)$ and

$$p(\theta) = \frac{1}{\mu(\theta, \mathbb{R}_+)} \int_{\mathbb{R}_+} p\mu(\theta, dp).$$

By construction,

$$U^s(\theta, \theta') = \pi(\theta')(p(\theta') - v^s_\theta).$$

Moreover, if (μ, λ) satisfies market clearing, then

$$U^b = \sum_{\theta \in \Theta} f_\theta \int_{\mathbb{R}_+} (v^b_\theta - p)\mu(\theta, dp) = \sum_{\theta \in \Theta} \pi(\theta)(v^b_\theta - p(\theta)).$$

Thus, (π, p) is incentive compatible, individually rational, and budget balanced if (μ, λ) is incentive compatible, individually rational, and satisfies market clearing.

Now consider a mechanism (π, p), not necessarily deterministic, and define the matching mechanism (μ, λ) to be such that: (*i*)

$$\mu(\theta, P) = \pi(\theta) \int_{\mathbb{R}_+} \mathbb{I}_P(p) dF_\theta(p),$$

where $\mathbb{I}_P$ is the indicator function of $P \subseteq \mathbb{R}^*$ and F_θ is the cumulative density function of $p(\theta)$; and (*ii*) $\lambda(\theta, P) = f_\theta \mu(\theta, P)$. By construction, the matching mechanism (μ, λ) satisfies market clearing. Notice that if $f = \sum_i \alpha_i \mathbb{I}_{P_i}$ is a simple function, then

$$\int_{\mathbb{R}_+} f(p)\mu(\theta, dp) = \sum_i \alpha_i \pi(\theta) \int_{\mathbb{R}_+} \mathbb{I}_{P_i}(p) dF_\theta(p) = \pi(\theta) \int_{\mathbb{R}_+} f(p) dF_\theta(p).$$

Hence,

$$U^s(\theta, \theta') = \int_{\mathbb{R}_+} (p - v^s_\theta)\mu(\theta', dp) = \pi(\theta) \left(\mathbb{E}[p(\theta')] - v^s_\theta\right).$$

Moreover, using the definition of λ,

$$U^b = \sum_\theta f_\theta \int_{\mathbb{R}_+} (v^b_\theta - p)\mu(\theta, dp) = \sum_\theta f_\theta \pi(\theta) \left(v^b_\theta - \mathbb{E}[p(\theta)]\right).$$

Thus, (μ, λ) is incentive compatible and individually rational if (π, p) is incentive compatible, individually rational, and budget balanced. This establishes the desired result.