Re-using collateral

A General Equilibrium Model of Rehypothecation∗

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Abstract

This paper considers an economy where agents face limited commitment and must transfer a durable asset as collateral to borrow. Rehypothecation allows lenders to re-sell this pledged asset or re-use it as collateral to sustain further borrowing. As such, it increases the ultimate supply of durable asset in the economy along collateralized credit chains. However, limited commitment now affects re-users since they effectively borrow the durable asset, limiting these benefits. I show that re-using collateral does not increase efficiency over an appropriate complete market benchmark. When some fundamental securities are missing, rehypothecation delivers strict welfare gains by relaxing collateral constraints. These gains appear significant in segmented markets where collateral scarcity issues are more acute. Finally, I show that rehypothecation also generates fragility along credit chains when aggregate leverage is high.

Keywords : Limited Commitment, Collateral Constraints, Rehypothecation, GE.

JEL codes : D41, D53.

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1 Introduction

In credit markets, lenders frequently require borrowers to post collateral as a protection against default. Households pledge their house as collateral for a mortgage while banks use financial assets such as government securities. While households remain the owners of their house during the transaction, financial intermediaries can arrange for the legal transfer of the asset used as collateral. This practice, commonly known as rehypothecation, allow lenders to sell or re-use the collateral for their own funding. While rehypothecation may seem bizarre, the ISDA\(^1\) reports that borrowers grant such rehypothecation rights in 73.7% of trades surveyed. With rehypothecation, collateral circulates through credit chains. These chains also generate risk as borrowers may be wary of losing access to their re-used asset. On account of those risks, Canadian law prohibits rehypothecation and several pending reforms in the US and the EU\(^2\) effectively prevent re-use. The current debates relies mostly on qualitative assessments and the literature does not yet offer a formal analysis of rehypothecation. The mere size of collateralized financial markets where rehypothecation is common practice (derivatives alone represent $700 trillions of gross notionals) thus motivates this analysis.

This paper provides a model to account for the trade-off between circulation of collateral and re-use risk in a general equilibrium framework. My theory of rehypothecation builds on Geanakoplos (1996) where a durable asset is used as collateral to secure short positions in financial securities\(^3\). I add rehypothecation to his model and allow lenders to re-use a fraction of the collateral received to secure their own short positions. With re-use rights, a lender effectively borrows the durable asset and faces limited commitment ex-post. This friction limits the ability of collateral to circulate without affecting ex-ante risk-sharing possibilities.

The technical contribution of this paper is to frame a model of rehypothecation as a simple extension of Geanakoplos (1996). Its first building block is the distinction be-

\(^1\)International Swaps and Derivatives Association, 2013 Margin Survey

\(^2\)Section VII of the Dodd-Frank Act on central clearing of swaps implies that collateral cannot be rehypothecated. Similarly, dealer banks now have to obligation to notify their clients that their collateral can be segregated. The EU’s own EMIR regulation introduces similar requirements for cleared swaps.

\(^3\)Some important models of default with limited commitment use a different technology than collateral to enforce promises. In Kehoe and Levine (1993), defaulting agents are excluded from financial markets. Dubey et al. (2005) uses non-pecuniary penalties for default. However, my work focuses precisely on the utilization of collateral, hence this modeling approach
tween segregated and unsegregated collateral. Only the latter may be re-used by traders. I also adapt the settlement mechanism to accommodate double sided limited commitment, resulting from ex-post default decisions. The main result of the paper establishes that rehypothecation proves redundant if agents may already trade collateral efficient securities without re-use. A precise combination of securities with low (and segregated) collateral requirements can deliver the same asset velocity as the original transaction with re-use. Hence, re-use is not a free lunch in this environment because (money) lenders become asset borrowers and face limited commitment. My analysis thus suggests that the implicit relationship between re-use and collateral efficiency in many recent studies should be revised.

The rest of the paper examines departures from the complete market benchmark to show how rehypothecation can strictly increase efficiency. With incomplete markets, rehypothecation helps freeing up collateral and relaxes collateral constraints. These gains appear larger with decentralized trading because intermediaries hold large gross positions and need more collateral. For instance, a bank may need to fund a loan it extends to an hedge fund by borrowing from a money market fund. With collateral constraints, intermediation mechanically doubles the need for pledgeable asset. Finally, I also show how re-use may generate fragility in the settlement process when many agents must return a scarce asset to their counterparty. Indeed, rehypothecation creates additional collateral out of a fixed quantity of pledgeable asset. The system fragility to external shocks depends on the strength of this multiplier effect.

Rehypothecation received much attention from policymaking circles as the 2007 financial crisis exposed risky collateral management practices at some dealer banks. Monnet (2011), Singh (2011) or Kirk et al. (2014) describe the trade-off between collateral circulation and collateral risk and provide rough measures of circulation at the aggregate or bank level. Bottazzi et al. (2012), account for re-use of securities through repurchase agreements but sidestep the commitment issue attached to returning securities. In Andolfatto et al. (2014), the value of a long-term relationship between traders mitigate the commitment issue and effectively substitutes for collateral. Their different focus on mon-

\[ \text{\footnotesize 4Several papers focusing on collateral demand induced by central clearing including Galbiati and Soramaki (2013) and Duffie et al. (2014) factor rehypothecation as an unambiguous gain for dealer banks.} \]

\[ \text{\footnotesize 5The 2011 failure of MF Global, a broker dealer who took bets on the European debt market re-using the collateral of its clients brought the practice to light.} \]
etary policy complements the analysis in this paper. Monnet and Nellen (2012) analyze different collateral segregation regimes with pure limited commitment. This paper combines the features from the contributions outlined above.

The need to re-use collateral suggests that pledgeable assets might be too scarce to sustain an efficient level of borrowing in the economy. Concerns about collateral scarcity gained prominence as Central Banks massive purchases of Treasuries reduce the supply of collateral-eligible assets, a point developed in Araújo et al. (2013). Re-use of collateral resembles other financial innovations such as tranching or pyramiding studied in Fostel and Geanakoplos (2012) or Gottardi and Kubler (2014) to economize on the pledgeable asset. Somewhat surprisingly, I show that in the absence of frictions, re-use fails to deliver the same efficiency gains. When markets are incomplete however, rehypothecation proves useful if pyramiding or tranching are not possible for legal or technical reasons. In practice, the widespread use of plain vanilla assets as collateral justifies rehypothecation over pyramiding.

At the aggregate level, rehypothecation contributes to the creation of credit chains resembling that of Kiyotaki and Moore (1997). As in their model, an unanticipated shock propagates through the system when agents hold large gross positions (cf Section 6). This effect is reminiscent of contagion in interbank networks as in Allen and Gale (2000) or clearing mechanisms a la Eisenberg and Noe (2001). Rehypothecation creates a new channel for contagion through the need to return the asset used as collateral. I show that a contract feature for settlement based on actual market practice mitigates these concerns.

Finally, rehypothecation stresses the analogy between collateral and money. As means of exchange, both determine the economy’s ability to trade efficiently in an environment with limited commitment. The concept of money velocity extends to collateral with rehypothecation as the same piece of asset can back several transactions. However, money ultimately replaces credit transactions with spot trades for exchanging goods. On the contrary, collateral fundamentally backs credit transactions for future state contingent commodities. Hence limited commitment is an issue for re-using collateral and limits the gains from rehypothecation.

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6 See CGFS (2013) for a recent take on the empirical debate around asset scarcity.
7 The ISDA 2013 Margin Survey reports that cash and government securities make up for more than 90% of collateral pledged by a sample of financial institutions.
The paper is organized as follows. In Section 2, I present a simple example to illustrate the main mechanisms at play with rehypothecation. Section 3 then introduces the general model and discusses its novel features. The main irrelevance result on rehypothecation appears on Section 4 while Section 5 shows how rehypothecation can restore efficiency in an incomplete market environment. The discussion about settlement and fragility with re-used collateral appears in Section 6. Finally, Section 7 concludes.

2 Re-using collateral: an example

This example illustrates how rehypothecation may facilitate trade and improve welfare in an environment where collateral is scarce. To avoid repetitions with the exposition of the model in Section 3, the argument is stripped down to its core features.

Example 1

Consider a two period \((t = 0, 1)\) competitive one good economy with 3 types of agents, \(i = A, B, C\). There are two states of the world in \(t = 1\) denoted \(s = 1, 2\) with equal probability \(\pi(s) = 1/2\). Agents are risk-averse, have identical VNM preferences and do not discount period 1 payoffs. Agent \(i\) endowment is denoted by \(e_i^0\) in period 0 and \(e_i^1(s)\) in period 1 and state \(s\). Let \(e \in (1, \infty)\). We have

\[
\begin{align*}
  e_A^0 &= e + 3/4 \\
  e_A^1(1) &= e - 1/2 \\
  e_A^1(2) &= e - 1 \\
  e_B^0 &= e + 1/4 \\
  e_B^1(1) &= e - 1/2 \\
  e_B^1(2) &= e \\
  e_C^0 &= e - 1 \\
  e_C^1(1) &= e \\
  e_C^1(2) &= e - 1
\end{align*}
\]

In addition, agent \(C\) is endowed with one unit of the a tree that delivers \(x(s) = s\) units of the consumption good in state \(s\) of period 1. With this endowment specification, there is no aggregate uncertainty in this economy. Agents may trade the tree and their holdings are denoted by \(\{\theta_i\}_{i=A,B,C}\). Agents may also trade in period 0 a bond in zero net supply with constant face value \(R(s) = 1\). A bond seller thus promises to pay 1 unit of consumption good to the buyer in period 1. Let \(\phi_i\) denote the quantity of bond held by agent \(i\). Agent \(i\)'s portfolio is thus given by the pair \((\theta_i, \phi_i)\).

Agents can default at no cost on promises to deliver goods but may use the tree as collateral to secure borrowing. Hence, a bond seller (i.e. a borrower) must post one
unit of the tree as collateral for every bond unit sold. Observe that this is the minimum quantity to ensure repayment by a borrower in state $s = 1$. This friction leads to the following constraint on trades.

\[ \theta^i + \min\{0, \phi^i\} \geq 0 \quad (1a) \]

In the following, I refer loosely to (1a) as the collateral constraint of agent $i$. In particular, $\theta^i \geq 0$, i.e. agents cannot take short positions in the tree. This is natural in our environment with limited commitment as shorting the tree amounts to a non-credible promise to pay $x(s)$ in state $s$. In this example, rehypothecation precisely relaxes this constraint.

\textit{Symmetric Efficient Allocation}

In this example, total endowment is constant in every date/state and equal to $3e$. Because agents are risk-averse, consumption across states must be constant in a Pareto Efficient allocation. Hence, the symmetric efficient allocation has any agent consuming $c^* = e$ in every date/state.

\textit{Decentralization with rehypothecation}

Agents can trade two assets (the tree and the bond) with payoff matrix

\[ R = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \]

There are two states and two non-colinear assets, hence a unique portfolio which finances $c^*$

\[ (\theta^A, \phi^A) = (1/2, 0) \quad (\theta^B, \phi^B) = (-1/2, 1) \quad (\theta^C, \phi^C) = (1, -1) \]

Agent $B$’s portfolio violates collateral constraint (1a) which prevents tree short-selling. I show that this is no longer the case if agents may re-use $1/2$ and no more than $1/2$ of the tree pledged as collateral. Observe indeed that agent $B$ would receive one unit of the tree as collateral to secure agent $C$’s short position in the bond. Rehypothecation allows him to sell $1/2$ of the tree to agent $A$. Hence, agent $B$ simultaneously lends cash against collateral to $C$ but also borrows $1/2$ units of collateral to short sell. Formally, constraint
(1a) is changed into
\[ \theta^i + \frac{1}{2} \max\{0, \phi^i\} + \min\{0, \phi^i\} \geq 0 \]
which holds for \((\theta^B, \phi^B) = (-1/2, 1)\). This pattern of trade implements the allocation \(e^*\).

Agent \(C\)'s incentives to repay do not change with respect to the no-rehypothecation case. However, since \(B\) has effectively borrowed the collateral (to short sell), he may fail to turn it back. Indeed, he only holds 1/2 units after the sale to agent \(A\). To deliver back all the collateral posted by \(C\), he must buy back (from \(A\)) the quantity he re-used. Hence, with rehypothecation, agent \(B\) also faces a strategic default decision. If he defaults, agent \(C\) obligation to him is cancelled and he gets a payoff of 0. If he buys back the 1/2 units at price \(x(s) = s\), he can deliver back the collateral to \(C\) and nets a gain of \(1 - 1/2x(s) \geq 0\), i.e. the bond face value minus the cost or buying back the tree.

Limited re-use explains why rehypothecation may work in the presence of double sided limited commitment. This feature is crucial to the general analysis developed in Section 3. A lender returns the collateral he acquired not to lose the payment on the loan it secures. Two elements are crucial for this argument though

(i) First, upon failing on his obligation to turn back collateral, agent \(B\) loses not only the bond payment \(R(s) = 1\) but also the collateral he did not re-use. As we will see, a distinction between segregated and unsegregated accounts for collateral helps to implement this outcome consistently with the fundamental limited pledgeability friction.

(ii) Second, agent \(B\) is able to buy back (in period 1) the tree at a price equal to the service flow or dividend \(x(s)\). The exposition will prove that one can either sidestep this issue with a mild assumption or rationalize this feature with a well-defined settlement process.

In this example, agent \(B\) re-uses the collateral to sell it to agent \(A\). More generally, rehypothecation can help recycling collateral in order to support economically valuable trades. This encompasses re-sale but also re-pledge, i.e. using the pledged collateral to secure additional borrowing in the same or different debt instruments. However, the example suggests that agents who re-use collateral must be given proper incentives to return it to their counterparties. This friction limits the economic value of rehypothecation. The rest
of the paper is concerned mainly with identifying environments where re-use may bring about strict welfare gains, as in this example.

### 3 The Model

#### 3.1 Physical Environment

Consider a pure exchange economy with two periods \( t = 0, 1 \) and several states of the world \( s = 1, \ldots, S \) in period 1. The economy consists of a set of agents \( I = \{1, 2, \ldots, I\} \) and a perishable consumption good. Each type \( i \) represents a continuum of agents of mass 1. Agent \( i \) endowment of the consumption good is \( e^i = (e^i_0, e^i_1(1), \ldots, e^i_1(S)) \in \mathbb{R}^{S+1} \). Preferences are Von Neumann-Morgenstern over consumption streams. Function \( u^i \) denotes instantaneous utility so that agent \( i \)'s preferences over consumption bundle \((c_0, c_1)\) are given by

\[
U^i(c_0, c_1) = u^i(c_0) + \sum_{s=1}^{S} \pi(s)u^i(c_1(s))
\]

where \( \pi(s) \) denotes the probability that state \( s \) realizes in period 1. Utility functions \( u^i : \mathbb{R}_{++} \rightarrow \mathbb{R} \) are strictly monotone, \( C^2 \), strictly concave and verify the Inada condition: \( \lim_{c \to 0} u^i(c) = \infty \).

The economy is endowed with \( \theta_0 \) units of a tradable Lucas tree: a durable asset\(^8\) which delivers a quantity \( x(s) \) of consumption good in state \( s \in \{1, \ldots, S\} \) of period 1 but no dividend in period 0. Agent \( i \) initially holds \( \theta^i_0 > 0 \) units of the tree. Each agent has two different accounts called a **segregated** and a **non-segregated** account to handle his holdings of the tree. This distinction becomes effective when I describe securities in the financial environment. Agent \( i \) total endowment in state \( s \) is then given by \( \omega^i(s) = e^i_1(s) + \theta^i_0 x(s) \). The main friction of the model is that the first component of endowment, \( e^i_1 \), cannot be pledged. This means that that no agent can be liable beyond his holdings of the durable asset.

\( \footnote{There are actually two goods in the economy: the perishable consumption good and the tree. Here we assume that the intrinsic valuation for the tree depends only on the quantity of consumption goods (dividends) it yields in period 1. I relax this standard assumption which has consequences for rehypothecation in Appendix B} \)
3.2 Financial Environment

3.2.1 Securities

With limited pledgeability, agents must use the tree as collateral to borrow. Lenders can re-use the collateral they receive to secure their own borrowing. A financial security (or simply a security) is thus defined as follows:

**Definition 1:** A security $j$ is a triplet $(\bar{R}_j, \bar{k}_j, \alpha_j) \in \mathbb{R}_+^S \times \mathbb{R}_+ \times [0, 1]$ where $\bar{R}_j(s)$ is the promised amount to be paid in state $s$, $\bar{k}_j$ is the quantity of tree to be posted as collateral and $\alpha_j$ is the fraction of this collateral to be held in the buyer’s segregated account.

In the following, I call seller or borrower (resp. buyer or lender) the agent who sells (resp. buys) a security. My definition with rehypothecation collapses to the standard concept of a collateralized security of Geanakoplos (1996) and others when $\alpha_j = 1$. When $\alpha_j < 1$, the security grants rehypothecation rights to the buyer. The set of all tradable securities is $\mathcal{J}$. The transfers involved by the sale of a security $j$ are illustrated on Figure 1. The seller must then transfer $\bar{k}_j$ units of the tree as collateral, a fraction $\alpha_j$ of which is segregated$^9$. The buyer may use the $1 - \alpha_j$ fraction of non-segregated collateral to carry his own transactions. Formally, he thus borrows the non-segregated collateral and promises to turn it back. Since reneging on promises is costless in our environment, lenders may opportunistically fail to do so. Ultimately, rehypothecation introduces a double sided limited commitment (henceforth 2SLC) problem. To make a distinction, I

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$^9$Practically, the segregated collateral could be stored in a third party’s dedicated account. In the tri-party repo market, BNY Mellon and JP Morgan provide this collateral storage facility.
call **Failure** the action of a lender not turning back collateral\(^{10}\) as in Johnson (1997). The terminology **Default** is saved for the decision of the borrower not to repay. I now analyze default and failure pattern for securities in \(J\). The following assumption will prove helpful for this purpose.

**Assumption A**: In period 1, a delivery of \(m\) units of collateral is equivalent to delivering \(mx(s)\) units of consumption good, for any \(m \geq 0\).

When called to turn back collateral, a lender can instead deliver consumption good. Assumption A ensures that this substitution is payoff-equivalent for borrowers. Fundamentally, when collateral is repledged, all lenders may not be able to deliver simultaneously the tree. Hence, Assumption A facilitates settlement by giving more leeway to collateral receivers. In the Appendix, I show that Assumption A obtains as the natural reduced-form version of a sequential settlement process where agents must return the tree and not consumption good.

### 3.2.2 Default Resolution Mechanism (DRM)

I now characterize securities’ payoff provided that lenders may also fail to return collateral. To avoid uninteresting complication, I restrict the analysis to symmetric equilibria which require agents of the same type to follow the same default and fail pattern across securities. Furthermore, an agent either defaults (resp. fails) or pays (resp. delivers) but cannot randomize between the two strategies. Practically, this rules out partial default or failure. Let \((B, L)\) be a type-pair of borrower (seller) and lender (buyer) for promise \(j\). Denote \(d_B^j(s) \in \{0, 1\}\) (resp. \(f_L^j(s)\)) decisions to default (resp. fail) by each trader in state \(s\). The DRM maps \((d_B^j(s), f_L^j(s))\) to trader’s payoffs. Most importantly, it requires payment from borrowers (resp. delivery from buyers) only if the counterparty itself complies\(^{11}\).

\(^{10}\)In the narrower sense of Fleming and Garbade (2005), failure (to settle) would be a - non-default - delay in delivering the tree. Because such delays can matter when re-used collateral must be returned, I analyze this issue in Section 6.

\(^{11}\)This feature requires some coordination in the payment/delivery system. A plausible mechanism for implementation relies on two stages for settlement. In the first stage, short (resp. long) agents announce decision to default (resp. fail) on their positions. In the second stage, payments and collateral transfers are implemented through the DRM based on these decisions by an automaton.
Indeed, $B$ payoff $\Pi^B(d^B_j(s))$ is

$$\Pi^B(0) = f^L_j(s)\alpha_j\bar{k}_jx(s) + (1 - f^L_j(s))(\bar{k}_jx(s) - \bar{R}_j(s))$$
$$\Pi^B(1) = 0$$

where $f^L_j(s) \in \{0, 1\}$ is taken as given. Importantly if $B$ does not default ($d^B_j(s) = 0$) he effectively pays only if counterparty $L$ does not fail, $f^L_j(s) = 0$. In the legal parlance, collateral pledgors have a right to set-off when the receiver fails to comply with his obligation to return collateral.

Consider now the situation of agent $L$ who faces the following trade-off:

$$\Pi^L(0) = d^B_j(s)\bar{k}_jx(s) + (1 - d^B_j(s))\bar{R}_j(s)$$
$$\Pi^L(1) = (1 - \alpha_j)\bar{k}_jx(s)$$

where $d^B_j(s) \in \{0, 1\}$ is taken as given. If $L$ announces that he does not fail ($f^L_j(s) = 0$), he receives the security face-value when $B$ complies or (all) the collateral if $B$ defaults. When failing ($f^L_j(s) = 1$), he walks away with the non-segregated part of collateral but misses the payment $\bar{R}_j(s)$ from the security. With the natural substitutability built in the DRM, an agent finds it optimal to comply when the counterparty does not default and failure occur simultaneously. To put it otherwise, this DRM allocates the tree across all contingencies without ambiguity.

Default occurs when the face value exceeds the collateral value (as in the 1SLC model). Failure occurs when the face value falls short of the non-segregated fraction of collateral. In other contingencies, payment and delivery follow contractual obligations. We can thus

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11 It thus avoids undesirable outcomes whereby either (i) payments remain un-allocated or (ii) default and failure occur simultaneously. To put it otherwise, this DRM allocates the tree across all contingencies without ambiguity.
write the actual payoff of a security $R_j = (R_j(1), ..., R_j(S))$ as

$$R_j(s) = \begin{cases} (1 - \alpha_j)\bar{k}_j x(s) & \text{if } \bar{R}_j(s) < (1 - \alpha_j)\bar{k}_j x(s) \\ \bar{R}_j(s) & \text{if } \bar{R}_j(s) \in [(1 - \alpha_j)\bar{k}_j x(s), \bar{k}_j x(s)] \\ \bar{k}_j x(s) & \text{if } \bar{R}_j(s) > \bar{k}_j x(s) \end{cases}$$

(2)

Observe that all three components of a security matter to determine the actual payoff but not the identity of the traders. Consider in particular the role of the re-use parameter $\alpha_j$. Partial segregation creates a wedge between the default threshold for a borrower and the failure point for a lender. With complete segregation ($\alpha_j = 1$), equation (2) boils down to $R_j(s) = \min\{\bar{R}_j(s), \bar{k}_j d(s)\}$ as only borrowers face a strategic choice. This is the standard expression for a collateralized security payoff found in the literature. Alternatively, without segregation ($\alpha_j = 0$), the payoff is given by $R_j(s) = \bar{k}_j x(s)$. The transaction is exactly a sale of the tree "pledged" as collateral. When $\alpha_j$ is interior, security $j$ is a combination of a sale of $(1 - \alpha_j)\bar{k}_j$ units of the tree and a residual security $j'$ with payoff $R_{j'}(s) = R_j(s) - (1 - \alpha_j)\bar{k}_j x(s)$. This observation provides the intuition for the spanning result in Lemma 2.

I denote $\mathcal{E}(\mathcal{J})$ the economy where the set of agents $\mathcal{I}$, preferences $\mathcal{U} := \{u^i\}^i \in \mathcal{I}$ and endowments $\mathcal{W} = \{e^i, \theta^i_0\}^i \in \mathcal{I}$ are fixed. Securities in $\mathcal{J}$ are traded competitively in a centralized market. The matrix $R \in \mathcal{M}_{(S) \times \mathcal{J} + 1}$ collects the payoffs of securities $\mathcal{J}$.

### 3.3 Agent’s Optimization Problem and Equilibrium

**Consumer Problem**

Denote $\theta^i$ agent $i$’s position in the market for the tree. Since short security positions need to be backed up by collateral, it is useful to distinguish purchases $\phi_j^+ \geq 0$ and sales $\phi_j^- \geq 0$ of security $j \in \mathcal{J}$. Prices of collateral and securities are respectively $p \in \mathbb{R}_+$ and $q \in \mathbb{R}_+^\mathcal{J}$. The problem of agent $i$ given asset prices and returns - defined by equation (2) - can
be written

\[
\max_{\{c, \theta, \phi\}} \ u^i(c_i^0) + \sum_{s \in S} \pi(s) u^i(c_i^1(s)) \quad (3)
\]

s. to

\[
c_i^0 + \sum_{j \in J} q_j \phi_j^{i+} + p \theta^i \leq e_i^0 + p \theta_0^i + \sum_{j \in J} q_j \phi_j^{i-} \quad (4)
\]

\[
c_i^1(s) + \sum_{j \in J} \phi_j^{s-} R_j(s) \leq e_i^1(s) + \theta^i x(s) + \sum_{j \in J} \phi_j^{s+} R_j(s) \quad \forall s \in S \quad (5)
\]

\[
\theta^i + \sum_{j \in J} (1 - \alpha_j) \bar{k}_j \phi_j^{i+} - \sum_{j \in J} \bar{k}_j \phi_j^{i-} \geq 0 \quad (6)
\]

Equations (4)-(5) are standard pieces of a financial market budget constraint. Rehypothecation affects the collateral constraint (6) as an agent can re-sell or re-pledge a fraction of the collateral he receives as a lender. In particular, agents may effectively hold a negative position in the tree market \( \theta^i < 0 \) if they receive collateral through long positions. This feature reminds of Bottazzi et al. (2012) where agents cannot issue securities but are able to sell them after they borrowed it. The collateral constraint with rehypothecation is similar in spirit to their “Box constraint”.

**Equilibrium Definition**

A symmetric equilibrium of economy \( E(\mathcal{J}) \) is a feasible allocation \( (c_0, c_1) \in \mathbb{R}^{(S+1) \times I} \), a price vector \((p, q) \in \mathbb{R}^{J+1}_+ \) such that \( \forall i \in I \), \((c_i^0, c_i^1)\) solves agent \( i \)'s optimization problem (3)-(6) and securities market clear, i.e.

(Feasibility) \[ \sum_{i=1}^I c_i^0 \leq \sum_{i=1}^I e_i^0, \quad \sum_{i=1}^I c_i^1(s) \leq \sum_{i=1}^I e_i^1(s) + \theta_0 x(s) \quad \forall s = 1..S \quad (7) \]

(Market Clearing) \[ \sum_{i=1}^I \theta^i = \theta_0, \quad \sum_{i=1}^I (\phi_j^{i+} - \phi_j^{i-}) = 0, \quad \forall j = 1..J \quad (8) \]

**Proposition 1**: Economy \( E(\mathcal{J}) \) admits a competitive equilibrium.

**Proof**: \( \triangleright \) Given the regularity conditions imposed on utility functions, the existence
of equilibrium follows as in the standard 1SLC environment. This result can be seen as a particular case of Theorem 1 in Geanakoplos and Zame (2013). The result would hold in most 2 period financial market models with any linear constraint (this is the case of collateral constraint 6). Hence, the contribution here consists in setting a general equilibrium model of rehypothecation in a familiar environment to exploit standard results. Existence of equilibrium does not require any new technique, hence the outside reference for the proof.

3.4 Discussion of the Set-up

Default and Failure

In my set-up, because of limited pledgeability, borrowing is non-recourse. Private debt is defaulted upon when its face value exceeds that of collateral. Symmetrically, borrowing collateral leads to failure when the repayment falls short of the value of unsegregated collateral. Such events are quite rare in financial markets. As the exposition will make clear, this theory does not depend and there being default or failure in equilibrium. It simply acknowledges restrictions on trading imposed by limited commitment. In practice, other forces such as reputation costs, penalties or market exclusion might discipline borrowers and lenders. However these costs would reduce the need for posting collateral in the first place. Hence, the desirability of rehypothecation should be independent from these alternative disciplining devices. For simplicity, I assume away such costs in the analysis.

Rehypothecation Rights

In my model, rehypothecation rights can be specified on a security basis. In practice, these rights are limits set in Master Agreements and by regulators. For instance, rehypothecation is banned in Canada. In the US, brokers/dealers may re-use an amount up to 140% of the debit balance 13 of their client under Regulation T. Under EU rules, counterparties actually bargain over the amount eligible for re-use. Save for the possibility to bargain, my framework can fit these provisions with a lower bound $\bar{\alpha}$ on $\alpha_j$ imposed on the whole financial structure. Because borrowing or collateral is risky and can lead to

\footnote{See SEC Rule 15c3-3. The debit balance is an equilibrium object in our model hence the different formulation as a fraction of total collateral posted.}
failure, agents would not necessarily push for higher limits on re-usable collateral.

Assumption A: Cash for collateral

Rehypothecation rights allow a lender to re-use the non-segregated collateral. At settlement date in period 1, the total collateral pledged might exceed the physical amount of tree available. Returning the tree to settle all trades simultaneously would prove unfeasible. Assumption A neutralizes this concern as agents may substitute the consumption good for collateral\textsuperscript{14} in period 1. Still, settlement could take place sequentially so that collateral is progressively freed when trades unwind. Appendix A introduces this orderly settlement process to rationalize Assumption A. The fact that this assumption favors rehypothecation should actually give more weight to the irrelevance result of Section 4.

The next section presents the main irrelevance result of the paper which states that rehypothecation is redundant when markets are complete in a sense made precise below. The result extends to decentralized trading.

4 Rehypothecation and Collateral Scarcity

In this model, agents trade in order to share risk and smooth consumption over time. The ability to conduct these trades might be limited by market incompleteness as in the standard GEI literature. More importantly, when agents face collateral constraints, the availability of the tree proves crucial for efficiency. In the extreme case where the economy has no tree, $\bar{\theta}_0 = 0$, agents cannot commit to repay any debt and stay in autarky. Singh (2011)’s emphasis on the link between rehypothecation and collateral velocity suggests that re-using collateral allows to save on a potentially scarce resource and ultimately improves market efficiency. The rest of this paper analyzes this claim and characterizes financial environments where it holds.

Proposition 2, the main result of this paper, plays down the importance of rehypothecation.\textsuperscript{15}

\textsuperscript{14}In practice, some Master Agreements contain provisions with the flavor of Assumption A. This is the case of the English Credit Support Annex to the ISDA Master Agreement which only bind receivers to return “equivalent” property. Counterparties may thus agree on the delivery of a different security than that pledged initially. Assumption A offers a similar flexibility to trading partners. See Monnet (2011) for a comparison of the different Credit Support Annexes. Unfortunately, the ISDA does not provide detailed numbers on the use of each of this Annex.
ecation. Indeed, I show that without restrictions on security design, agents are equally well-off trading securities without rehypothecation rights. In other words, welfare gains from collateral re-use may only materialize when some financial securities are missing.

4.1 Replicating Financial Structure

Our claim that rehypothecation does not fundamentally increase collateral availability relies on a spanning argument. Given a financial structure \( \mathcal{J} \), we want to find a “rehypothecation-free” financial structure \( \mathcal{J}_1 \) that can substitute for securities in \( \mathcal{J} \). The following definition formalizes the concept of replication in our environment with collateral constraints.

**Definition 3**: A financial structure \( \mathcal{J}_1 \) replicates another structure \( \mathcal{J} \) iff \( \forall j \in \mathcal{J} \), \( \exists \mathcal{J}_1(j) \subset \mathcal{J}_1 \), \((\theta, \phi) \in \mathbb{R}_+ \times \mathbb{R}_+^{|\mathcal{J}_1(j)|}\) such that

(i) \( \forall s \in S, \quad \theta x(s) + \sum_{j_1 \in \mathcal{J}_1(j)} \phi_{j_1} R_{j_1}(s) = R_j(s) \)

(ii) \( \theta + \sum_{j_1 \in \mathcal{J}_1(j)} \phi_{j_1} \bar{k}_{j_1} \leq \bar{k}_j \)

(iii) \( \sum_{j_1 \in \mathcal{J}_1(j)} \alpha_{j_1} \phi_{j_1} \bar{k}_{j_1} \leq \alpha_j \bar{k}_j \)

\((\theta, \phi)\) is called a replicating portfolio and we denote \( \mathcal{J}_1 \in Sp(\mathcal{J}) \)

In words, for every security of \( \mathcal{J} \), there must be a portfolio made of the tree and securities in the set \( \mathcal{J}_1 \) that delivers the same payoff. This is the standard spanning feature of a replicating portfolio. In addition, selling the replicating portfolio does not require more collateral than the original security. Finally, the replicating portfolio should not imply more collateral segregation. Criteria \((ii)\) and \((iii)\) acknowledge the role of the tree as collateral and recognizes the importance of tree availability or unencumbered collateral to carry trades. Clearly \( Sp(\mathcal{J}) \neq \emptyset \) since \( \mathcal{J} \in Sp(\mathcal{J}) \).

The propositions of this section will build on the following lemma:

**Lemma 1**: Let \( \mathcal{J} \) and \( \mathcal{J}_1 \) be such that \( \mathcal{J}_1 \in Sp(\mathcal{J}) \). Then, any equilibrium allocation of \( E(\mathcal{J} \cup \mathcal{J}_1) \) is an equilibrium allocation of \( E(\mathcal{J}_1) \).
The proof is in Appendix C. Lemma 1 can be seen as an extension of Proposition 2 in Araújo et al. (2012) for rehypothecation. There is no loss of generality in restricting trades to a replicating financial structure. Intuitively, we can substitute every security of \( \mathcal{J} \setminus \mathcal{J}_1 \) traded in equilibrium by a replicating portfolio. The latter delivers the same payoff. Requirements (ii) and (iii) of Definition 3 ensure that the substitution does not affect collateral availability. That a financial structure may substitute for another is of limited interest unless the former entails some genuine restrictions. Perhaps surprisingly, I show in Proposition 2 that agents need not trade securities with rehypothecation rights.

4.2 An Irrelevance Result

Allowing for rehypothecation has two distinct effects. First, as lenders may fail and keep the unsegregated collateral, these state-contingent decisions might create new securities that correspond better to agent’s trading needs. This observation mirrors Dubey et al. (2005)’s remark about the welfare impact of borrower’s default and needs not further emphasis. We rather focus on the popular claim that rehypothecation improves collateral availability through higher velocity. To analyze this argument in our environment, we compare an unrestricted financial structure to one without rehypothecation.

Consider then the unrestricted set of securities \( \mathcal{J}_0 \) together with a subset of \( \mathcal{J}_* \subset \mathcal{J}_0 \) of securities without rehypothecation rights.

\[
\mathcal{J}_0 = \left\{ (\bar{R}_j, \bar{k}_j, \alpha_j) \in \mathbb{R}^S_+ \times \mathbb{R}^+ \times [0, 1] \right\}
\]

\[
\mathcal{J}_* = \left\{ (\bar{R}_j, \bar{k}_j, \alpha_j) \in \times_{s=1..S}\{0, x(s)\} \times \{1\} \times \{1\} \right\}
\]

Observe that \( \mathcal{J}_* \) contains but is not equal to the set of (collateralized) Arrow Securities. This arises because of security-specific collateral constraints. Suppose indeed that \( S = 4 \) and \( x(s) = 1 \) for all \( s \) and that an agent wants to sell the payoff \((0, 0, 1, 1)\). With Arrow securities only, he needs to short \((0, 0, 1, 0)\) and \((0, 0, 0, 1)\) hence requiring 2 units of collateral. Only 1 is necessary if the security is available.\(^{15}\) The next Lemma show

\(^{15}\)This argument also makes clear that with ex-post (or equivalently portfolio wide) collateral constraints as in Chien et al. (2011), Arrow Securities do suffice. For Credit Default Swaps (CDS) transactions, initial margins are a mix of both. Still, I consider that security-specific collateral constraints...
that the no-rehypothecation structure $\mathcal{J}_*$ replicates the unrestricted financial structure $\mathcal{J}_0$.

**Lemma 2**: $\mathcal{J}_* \in \text{Sp}(\mathcal{J}_0)$.

The proof adapted from Kilenthong (2011) is in Appendix C. Lemma 1 thus shows that for spanning purposes and collateral utilization, a smaller set of securities without rehypothecation suffices.\(^\text{16}\) Intuitively, a transaction with rehypothecation can be broken down into two components: the collateral loan and the residual payoff. To see this, let us consider a generic security $j = (\bar{R}_j, \bar{k}_j, \alpha_j)$. The actual payoff $R_j$ verifies

$$R_j(s) = (1 - \alpha_j)\bar{k}_j x(s) + \left[ R_j(s) - (1 - \alpha_j)\bar{k}_j x(s) \right]_{R_{j2}(s)}$$  \hspace{1cm} (9)

The first term on the RHS reflects the collateral loan feature of the transaction. It can be replaced by a sale of $(1 - \alpha_j)\bar{k}_j$ units of tree. The second term is the residual payoff $R_{j2}(s) \in [0, \alpha_j\bar{k}_j x(s)]$. If there exists an actual no re-use security $j_2$ with this payoff and collateral requirement $\alpha_j\bar{k}_j$, we can dismiss the original security $j$. Indeed, the replication uses the same amount of tree $\bar{k}_j$ and also leaves $(1 - \alpha_j)\bar{k}_j$ units of the tree available for further use. The tree that was lent with security $j$ is now sold in the replicating transaction. If such operation is feasible for any security $j$, rehypothecation becomes redundant. The proof of Lemma 2 shows how to realize this substitution for any generic security thanks to the no re-use securities in $\mathcal{J}_*$

We can now state the main result of this paper. The complete financial structure without rehypothecation rights $\mathcal{J}_*$ can deliver the same outcomes as the unconstrained structure $\mathcal{J}_0$.

**Proposition 2** Any equilibrium allocation of economy $\mathcal{E}(\mathcal{J}_0)$ is an equilibrium allocation of $\mathcal{E}(\mathcal{J}_*)$.

The proof follows directly from Lemma 1 and 2. Proposition 2 reads as a negative statement. Rehypothecation does not allow to finance welfare improving allocations when

---

\(^{16}\)It can be shown that subsets of $\mathcal{J}_*$ containing only $2^{S-1} - 1$ securities also span $\mathcal{J}_0$ but this is of limited interest to our analysis.

better approximate market practice in general.
the standard securities in $\mathcal{J}$ are available. Re-use frees up collateral to support additional trades. However, the implicit asset loan may be defaulted upon by the collateral receiver in an environment with limited commitment. This affects the security payoff ex-post and possibility for risk-sharing ex-ante. Proposition 2 shows that the first effect may dominate the second only when some no re-use securities of $\mathcal{J}$ are missing.

At a fundamental level, Proposition 2 establishes that rehypothecation does not allow for welfare gains over standard security specific margins without re-use. This is an interesting, albeit disappointing result, because some feasible allocations may not obtain with such collateral rules. Asset specific margins indeed entail two types of restriction, given the fundamental pledgeability friction of our environment. First, a given piece of collateral cannot back several securities (no tranching). Second, only the tree itself, not financial securities, can serve as collateral (no pyramiding). Gottardi and Kubler (2014) precisely show that pyramiding or tranching realize the efficient use of the tree as collateral\textsuperscript{17}. Hence, Proposition 2 proves additionally that rehypothecation ranks below both these techniques as a collateral saving device. Proposition 2 thus leaves us with a puzzle given the popularity of rehypothecation compared to those alternative techniques. The difference between pyramiding and rehypothecation might seem surprising given the similarities between both techniques. In the former case however, any loan with payoff $\bar{R}_j(s) \leq \bar{k}_j x(s)$ backed by the tree may be (re)-used as collateral. With rehypothecation, only proportional re-use of the tree (a fraction $(1 - \alpha_j)\bar{k}_j$) is possible. Because it allows for collateral re-use in a much finer way, pyramiding dominates rehypothecation.

4.3 Robustness to Market Segmentation

Many financial instruments that require posting collateral like swaps or derivatives trade Over The Counter (OTC). For this reason, large dealer banks typically sell insurance through Credit Default Swaps to customers to later offset the position with a protection seller. Since end-users cannot trade directly, agents would benefit from the ability to

\textsuperscript{17}We refer the reader to this paper for a formal definition of these concepts as well as that of a feasible allocation with limited pledgeability. In our two periods environment, the additional feasibility requirement is that an allocation must verify

$$\forall i \in \mathcal{I}, \forall s \in \mathcal{S}, \ e^i_1(s) \geq e^i_1(s)$$

Intuitively, consumption cannot fall below the non-pledgeable endowment $e_i$
re-use the collateral to avoid double margining. I show to the contrary that centralized trading does not drive the equivalence result as long as securities in the spanning set \( \mathcal{J}^* \) may be (competitively) traded. While both market segmentation and missing financial securities entail some form of market incompleteness, only the latter affects the irrelevance result of Proposition 2. This should not appear surprising since Lemma 1 relies essentially on a spanning argument.

The rest of this section formulates the argument in a more formal way but does not involve additional difficulty. For the sake of comparison, I maintain the competitive nature of the model. My analysis thus overlooks price formation processes particular to decentralized markets. Market segmentation is captured by the assumption that some agents may not trade with others. Agents are now disposed on a connected graph \( G \). A symmetric adjacency matrix \( A \in \mathcal{M}_{I \times I} \) indicates which agents are connected to each other. If \( A(i, h) = A(h, i) = 1 \), agents \( i \) and \( h \) can trade together. Hence the set of agents \( I \) and the adjacency matrix \( A \) ultimately characterize the graph \( G := \{I, A\} \). Let \( A_{I_0} \) denote the adjacency matrix restricted to a subset \( I_0 \subseteq I \). Trading takes place on Local Markets which correspond to subgraphs of \( G \). My definitions of Local Markets (LM) reads as follows.

**Definition 4** A set of LM is a finite set of subgraphs \( \mathcal{H} = \{H_n := (I_n, A_{I_n})\}_{n=1..N} \) such that:

i) \( \mathcal{I} = \bigcup_{n=1}^{N} I_n \)

ii) \( \forall n = 1..N, H_n \) is complete.

iii) \( \forall n \neq n', H_n \nsubseteq H_{n'} \)

Point i) states that every agent participates in at least one LM and hence rules out exclusion from financial markets. Point ii) requires traders in a LM to be able to trade with each other. Finally, with iii), I impose that no local market is contained in another LM. This last assumption is not crucial but facilitates the interpretation of the LM concept. However, local markets may overlap and one may describe agents at the intersection of several local markets as intermediaries\(^{18}\). Observe that while two traders in distinct LM have access to different sets of counterparties, similar securities are available. Still,

\(^{18}\)In Section 5.2, agent \( B \) of Example 2 is an intermediary
local markets matter with collateral constraints because positions need to be cleared at the local level. This might increase the amount of collateral needed to realize a given consumption transfer across time or states\textsuperscript{19}. Such an economy is denoted by $\mathcal{E}(\mathcal{H}, \mathcal{J})$. The following Corollary shows that the irrelevance result extends to decentralized trading.

**Corollary to Proposition 2:** Every equilibrium allocation of (decentralized) economy $\mathcal{E}(\mathcal{H}, \mathcal{J}_0)$ is an equilibrium allocation of (decentralized) economy $\mathcal{E}(\mathcal{H}, \mathcal{J}_*)$

**Proof** ▶ The Corollary holds since the proof for Proposition 2 itself did not use that trading was centralized. In particular, the argument can be reproduced for every local market in $\mathcal{H}$ without any other modification than to the notation. ◀

As long as markets are competitive and the set of securities $\mathcal{J}_*$ is available, the result carries on with decentralized trading. Market segmentation does not affect the ability of financial structure $\mathcal{J}_*$ to support trades with an efficient use of collateral. Still, as I highlighted in the introduction, collateral arrangements are quite common in secured financial markets, suggesting that there should be economic value to rehypothecation. The analysis of this Section made clear that potential gains from rehypothecation in this environment\textsuperscript{20} may only materialize when markets are incomplete. In the next section, I take this friction seriously and show why rehypothecation can then strictly increase welfare. Interestingly, such gains appear to be larger with decentralized markets, as introduced in this section.

5 **Rehypothecation under Incomplete Markets**

Rehypothecation is redundant with complete markets. Proposition 2 established that trading in a complete financial structure $\mathcal{J}_*$ obviates the need for collateral re-use. There exists a number of reasons why such a complete set of securities might not be available\textsuperscript{21}.

\textsuperscript{19}This observation will prove important for the analysis of rehypothecation in centralized vs. decentralized (incomplete) markets of Section 5
\textsuperscript{20}In appendix B, I extend the model to allow for heterogeneous valuation for the collateral and suggests that the irrelevance result under complete markets also holds with this modification
\textsuperscript{21}Market incompleteness is exogeneous in my model since I do not model the supply side for financial securities. Private information about aggregate or idiosyncratic outcomes sometimes justify why some
This section shows that when collateral efficient financial securities are missing, rehypotheca
tion may be strictly desirable. To do so, I revisit first Example 1 where markets are
indeed incomplete. Then I show with Example 2 that the gains from re-using collateral
are typically larger with decentralized markets.

5.1 Analyzing Example 1

Let us remind the setting of Example 1 introduced in Section 2. There are 3 types of
agents, \(i = A, B, C\). and two states of the world in \(t = 1\) denoted \(s = 1, 2\) with equal
probability \(\pi(s) = 1/2\). Agents have identical VNM preferences and do not discount
period 1 payoffs. Let \(e \in (1, \infty)\) be given. Endowments are as follows.

\[
\begin{align*}
  e^A_0 &= e + 3/4 \\
  e^A_1(1) &= e - 1/2 \\
  e^A_1(2) &= e - 1 \\
  e^B_0 &= e + 1/4 \\
  e^B_1(1) &= e - 1/2 \\
  e^B_1(2) &= e \\
  e^C_0 &= e - 1 \\
  e^C_1(1) &= e \\
  e^C_1(2) &= e - 1
\end{align*}
\]

In addition, agent \(C\) is endowed with one unit of the a tree that delivers \(x(s) = s\) units of
the consumption good in state \(s\) of period 1. With our notation \(\theta^C_0 = 1\) while \(\theta^A_0 = \theta^B_0 = 0\).
As we highlighted before, the symmetric Pareto efficient allocation \(e^*\) features constant
consumption across agents, time and states equal to \(e\).

Complete Markets

Suppose two Arrow securities \(j_1, j_2\) are available to trade. Security \(j_s\) pays 1 in state
\(s\) and zero in the other state and requires \(1/s\) units of collateral. Define \(R_C\) (where \(C\)
stands for complete) the matrix of payoff :

\[R_C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}\]

where the first column contains the tree payoff. Period 1 transfers implied by allocation

---

securities are not traded. In the first case, DeMarzo and Duffie (1999) and more recently Dang et al.
(2012) show that debt-like securities are desirable because of their low information sensitivity. For
the second case, there exists a long tradition of Bewley models where idiosyncratic shocks are private
information and hence uninsurable. Even without such frictions, Carvajal et al. (2012) prove that security
designers might find it optimal not to complete the market.
**c** may be decentralized with portfolios \{((\theta^i, \phi^i_{j_1}, \phi^i_{j_2}))\}_{i=A,B,C} if

\[
\sum_i \theta^i = 1
\]

For \(s = 1, 2\)
\[
\sum_i \phi^i_{j_s} = 0
\]

\(\forall i \in \{A, B, C\}, \begin{bmatrix} e \\ e \end{bmatrix} = \begin{bmatrix} e^i_{1}(1) \\ e^i_{1}(2) \end{bmatrix} + R_{C_i} \cdot (\theta^i, \phi^i_{j_1}, \phi^i_{j_2}) \)

\[
0 \leq \theta^i + \min\{\phi^i_{j_1}, 0\} + \frac{1}{2} \min\{\phi^i_{j_1}, 0\}
\]

One can easily check that the following portfolios are a solution to the problem above:

\((\theta^A, \phi^A_{j_1}, \phi^A_{j_2}) = (1/2, 0, 0) \quad (\theta^B, \phi^B_{j_1}, \phi^B_{j_2}) = (0, 1/2, 0) \quad (\theta^C, \phi^C_{j_1}, \phi^C_{j_2}) = (1/2, -1/2, 0)\)

Finally, prices for the tree and the securities are equal to their state weighted payoff since the allocation exhibits perfect risk sharing and constant consumption. Thus we have \(p = 3/2\) and \(q_{j_1} = q_{j_2} = 1\). Arrow securities are collateral efficient because agents can diversify idiosyncratic risk with a minimal amount of the tree.

**Incomplete Markets : Bond Economy**

We now turn back to the incomplete (i.e. collateral inefficient) financial structure where agents can only trade the tree and a bond with face value equal to 1. The payoff matrix \(R_I\) (where \(I\) stands for incomplete) thus writes

\[
R_I = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}
\]

As shown in Section 2, granting rehypothecation rights on half of the collateral pledged for a bond allows to finance \(c^*\). The collateral constrain thus writes

\[
\forall i \in \{A, B, C\}, \quad \theta^i + \frac{1}{2} \max\{\phi^i, 0\} + \min\{\phi^i, 0\} \geq 0
\]

To finance \(c^*\), agent \(C\) borrows one unit from agent \(B\) who then sells half of the tree
obtained as collateral to agent \( A \). Agents thus hold the following portfolios

\[
(\theta^A, \phi^A) = (1/2, 0) \quad (\theta^B, \phi^B) = (-1/2, 1) \quad (\theta^C, \phi^C) = (1, -1)
\]

Fundamentally, agent \( B \) wishes to increase consumption by \((1/2, 0)\) in state 1 and 2 respectively. Under complete markets, he needs 1/2 units of Arrow Security, mobilizing 1/2 units of collateral: the tree agent \( C \) has to post. In the bond economy however, agent \( B \) must take a long position of 1 unit in the bond and sell 1/2 units of tree: this locks 1 unit of the tree as collateral. With rehypothecation, 1/2 units of this collateral can be re-used, leaving as before only 1/2 units segregated. Ultimately, with incomplete markets, rehypothecation frees up tree pledged as collateral for which segregation is not fundamentally required for risk-sharing. In fact, rehypothecation makes use of the slack in the lender’s limited commitment constraint which for a no re-use security \( j = (\bar{R}_j, \bar{k}_j, 1) \) writes as

\[
1 - \bar{\alpha}_j = \min_{s \in S} \frac{R_j(s)}{\bar{k}_j(s)x(s)}
\]

One can check that this number indeed equals 1/2 for the bond collateralized by one unit of the tree.

As we mentioned before, the irrelevance result illustrated with this example may not provide a positive theory of rehypothecation. One should not observe collateral re-use in the first place if some collateral efficient securities could be traded instead. Again, this analysis remains agnostic as to why markets would be incomplete in the first place. The purpose of this example was rather to show the positive effect of re-use given that such a friction exists.

### 5.2 Centralized vs. Bilateral Trading

This section emphasizes rehypothecation gains with decentralized trading. Again, given the equivalence results of Section 4, this analysis is meaningful only within an incomplete market environment. To anticipate on the results below, higher collateral needs with bilateral trade can justify rehypothecation, when it is not warranted in a centralized market.

**Example 2**

To account for decentralized trading, at least 3 agents \( i = A, B, C \) are needed. There
are again 2 states of the world. The asset pays off \( x(s) = s \) for \( s = 1, 2 \). Agents have the same utility function and do not discount period 1 payoffs. Let \( e \in (1, \infty) \) be given and consider the following endowments:

\[
\begin{align*}
\{ e_0^A &= e + 1/2 \\
 e_1^A(s) &= e - 1/2 \\
 \theta_0^A &= 0 \}
\end{align*}
\]

\[
\begin{align*}
\{ e_0^B &= e + 1/2 \\
 e_1^B(s) &= e - 1/2 \\
 \theta_0^B &= 0 \}
\end{align*}
\]

\[
\begin{align*}
\{ e_0^C &= e - 1 \\
 e_1^C(s) &= e + 1 - s \\
 \theta_0^C &= 1 \}
\end{align*}
\]

Given convex preferences and the absence of discounting, it is easy to see that the symmetric Pareto optimal allocation is \( c^*_i = c^* = e \) for all \( i = A, B, C \) and all \( t = 0, 1 \). Agent \( C \) wishes to borrow while agents 1 and 2 desire to lend. To put it otherwise, agents want to trade endowment across time and not across states as in 5.1. Hence, a non-contingent bond with face value \( R(s) = 1 \) appears as the perfect instrument. For borrowers not to default, it needs to be collateralized by one unit of asset so \( \bar{k} = 1 \). Finally, for lenders not to fail, rehypothecation should be limited by \( \alpha \geq 1/2 \), exactly as in the previous example.

**Centralized Trading**

With centralized trading, the bond decentralizes the efficient allocation and agents need not re-use collateral. For this, agent \( C \) borrows 1 while agents \( A \) and \( B \) both lend 1/2. This trades verifies agent \( C \)'s collateral constraint since \( \phi^C \leq \theta_0^C = 1 \). The allocation \( c^* \) is supported by price \( q = 1 \) for the bond given perfect risk sharing and absence of discounting. The inequalities above are also equivalent to the fundamental constraints on the limited pledgeability of income and no other trade pattern can improve upon this one.

**Decentralized Trading**

Suppose now that trading is bilateral as shown in figure 2 so that agent \( B \) qualifies as an intermediary. Precisely, to implement \( c^* \) agent \( B \) needs to lend 1 to agent \( C \) and borrow 1/2 from agent \( A \). Hence, he is still a net lender of 1/2 but he holds both long and short positions in the bond. The continuous (resp. dashed) curved arrows illustrate the transfer of cash (resp. collateral) in period 0.

Since 3/2 units are lent on aggregate, 3/2 units of collateral must be posted while there is only 1 unit of tree. With rehypothecation, agent \( B \) can re-use a fraction 1/2 of the collateral posted by agent \( C \) to borrow in turn from agent \( A \). While 3/2 units of
collateral are posted, total physical holdings of the tree still sum up to 1. After trading took place in period 0, we can decompose holdings between segregated and unsegregated account for every agent as follows:

\[
(\theta_A^S, \theta_A^U) = (1/4, 1/4) \quad (\theta_B^S, \theta_B^U) = (1/2, 0) \quad (\theta_C^S, \theta_C^U) = (0, 0)
\]

When borrowing from agent A, agent B recycles half of the collateral he receives from agent C. We are left to verify that agents who receive collateral do not want to fail on the tree loan they receive. As in the previous example, the no fail constraint writes

\[
\bar{R}_j(s) \geq \frac{1}{2} x(s) \iff 1 \geq \frac{1}{2} s
\]

which holds for \( s = 1, 2 \). In period 1, A thus delivers the 1/2 units he holds against the payment of 1/2 from B who can then return 1 unit of collateral to A against a payment of 1. Hence, the final holder of the tree is agent C as required by the optimal portfolio.

Bilateral trading implies double collateral posting for the intermediary (agent B) to finance the trade. This cost disappears in a centralized market where agents A and C can trade directly as explained before. On OTC markets, intermediaries typically hold both short and long positions in the same security to match trading needs of dispersed customers\(^{22}\). When collateral must be posted, the gross trade mechanically requires more collateral than the net transaction. Rehypothecation generates more collateral out of a given amount of tree to meet this increasing demand.

\(^{22}\) This is the essence of a matched-book repo trade when broker-dealers engage simultaneously in a repo with one customer and a reverse repo with another customer. Atkeson et al. (2013) also report that top dealer banks have large gross positions but small net positions in the CDS market.
6 Rehypothecation and Fragility

By definition, rehypothecation allows a lender to re-use a piece of collateral pledged by a borrower. If he exerts his right, the lender may not have the asset on his balance sheet upon settlement of their transaction. Assumption A, inspired by the English Credit Support Annex, neutralizes this concern as a cash payment may substitute for the physical asset delivery. However, some borrowers may not wish to see their collateral transformed and can opt for a different scheme, the New York Credit Support Annex.

In this case, the lender would purchase the asset in the market to make the contractual delivery. In my framework, Appendix A indeed shows that a market for the tree in period 1 allows agents to make up for their short positions before turning back the collateral. The analysis also stresses the importance of a well-behaved sequential process to ensure smooth settlement. Agents may not purchase back the tree simultaneously when the total collateral pledged exceeds the quantity of tree available in the economy, i.e:

\[\theta_0 < \sum_i \sum_j \bar{k}_j \phi_j^{-} \quad (10)\]

Inequality (10) involves endogenous variables rather than deep parameters. However, this condition typically holds in equilibrium since the purpose of rehypothecation is precisely to create more collateral out of a given quantity of pledgeable asset. Then, the need for a sequential settlement process described in Appendix A proceeds from a resource constraint on the availability of the tree.

The rest of the paper presents modifications to the original set-up which generate fragility specific to rehypothecation in the settlement process. Section 6.1 introduces discounting during settlement to create a trade-off between early and late delivery. I show that the trade-off works in favor of borrowers because they can postpone payment until they receive their asset back, a point emphasized in Fleming and Garbade (2005) for settlement fails. In Section 6.2, I discuss how exogenous default might propagate along credit chains when (10) holds. For simplicity, I introduce these shocks as zero probability events. Hence, the interest of this analysis rests on the amplification mechanism specific to rehypothecation.
6.1 Delivery Race for Collateral

In this section, we consider Example 2 and look at the impact of an unanticipated preference shock in period 1 which affects discounting between settlement sub-periods. Hence, the equilibrium positions derived in Section 5.2 are still valid. Remember that agent \( C \) borrows one unit of consumption good from agent \( B \), pledging 1 unit of tree with payoff \( x(s) = s \). In turn, agent \( B \) borrows \( 1/2 \) units from agent \( A \), re-using half of the collateral he received from agent \( C \). Hence, condition (10) holds with the left and right hand side respectively equal to 1 and \( 3/2 \).

Then, 2 rounds of settlement \( \tau = 1, 2 \) are necessary to clear all equilibrium positions. When the shock materializes, agents discount \( \tau = 2 \) payoffs at rate \( \delta < 1 \). Hence, with obvious notations, consumption in period 1 is given by

\[
c_i^1(s) = c_{1,1}^i(s) + \delta c_{1,2}^i(s)
\]

Agents may save the consumption good between \( \tau = 1 \) and \( \tau = 2 \) with net interest rate \( r = (1 - \delta)/\delta \). The tree pays its dividend in round \( \tau = 2 \) which is normalized to \( x(s)/\delta \). This ensures that the contribution of the tree to \( c_i^1(s) \) is still \( x(s) \) per unit whether it is sold in round \( \tau = 1 \) or consumed in period \( \tau = 2 \). Indeed, the natural price of the tree in round 1 would thus be \( p_{1,1}^*(s) = \delta \times 1/\delta x(s) = x(s) \). A possible interpretation of this preference shock is an unexpected increases of overnight rates on reserves from 0 to \( r > 0 \). Settlement sub-periods then correspond to two different business days. With discounting, the physical necessity to deliver over 2 rounds has real consequences unlike in Appendix A where \( \delta = 1 \).

We now introduce the definition of a settlement equilibrium given positions traded in period 0. Appendix A provides a more formal treatment.

**Definition:** A settlement equilibrium in state \( s \) is a pair of prices \( \{p_1, \tau(s)\}_{\tau=1,2} \), trades, default \( \{d_1, \tau(s)\}_{\tau=1,2} \) and fail \( \{f_1, \tau(s)\}_{\tau=1,2} \) decisions by borrowers and lenders respectively such that:

\[
i) \ \{d_1, \tau(s)\}, f_1, \tau(s)\}_{\tau=1,2} \text{ are optimal given } \{p_1, \tau(s)\}_{\tau=1,2}
\]

\[23\] Although relying on an unanticipated shock does not provide a fully satisfying answer, we observe that such a shock would have no consequence absent rehypothecation
ii) Tree market clears in every sub-period $\tau = 1, 2$

We now construct heuristically the settlement equilibrium when $\delta < 1$. Observe first that borrowers actually benefit from late delivery since the present value of collateral is $x(s)$ while the present value of the payment is $\delta < 1$. Hence, since borrowers did not default when $\delta = 1$, they do not when $\delta < 1$. Lenders, on the other side of the trade, lose with a late delivery which drives up the price of the asset in sub-period $\tau = 1$. I will now determine the price $p_{1,1}(s)$ of the tree in the first settlement round and whether lenders decide to fail or not. For this, one must equalize the value to deliver early and late for a lender. This writes

$$1 - 1/2p_{1,1}(s) = \delta - 1/2x(s)$$

To deliver in the first period, an agent must buy $1/2$ unit of tree at price $p_{1,1}(s)$. For an agent who holds the collateral, this is the opportunity cost of not selling it. Delivering in period 2 costs only $p_{1,2}(s) = x(s)$ but the security payment is discounted by $\delta$. In state $s = 1$, the clearing price is $p_{1,1}(s) = 3 - 2\delta > p^*_{1,1}(s)$. In state $s = 2$, the RHS is negative and lenders fail. One can then work out consumption in period 1 for all three agents

$$\begin{cases}
  c_A^1(1) = e \\
  c_A^2(2) = e
\end{cases} \quad \begin{cases}
  c_B^1(1) = e + 1 - \delta \\
  c_B^2(2) = e
\end{cases} \quad \begin{cases}
  c_C^1(1) = e - 1 + \delta \\
  c_C^2(2) = e
\end{cases}$$

Hence there is an ex-post transfer from agent $B$ to agent $C$ for re-using his collateral. This arises because collateral is scarce and all lenders cannot deliver in round 1. This does not affect agent $A$ who holds the collateral he needs to deliver before settlement starts. If $\delta$ is sufficiently close to 1, the benefits from collateral circulation should still overweight the settlement costs. Importantly, sequential delivery is necessary because condition (10) holds. Otherwise, all agents could settle positions in the first round of trading by buying the tree. Hence, the settlement costs associated to rehypothecation materialize only when the pledgeable asset is scarce\textsuperscript{24}.

\textsuperscript{24}This issue is often raised in the debate on rehypothecation. Terence Duffy, President of the Chicago Mercantile Exchange (CME) put it in a simple but compelling way during his MF Global House hearing: “if [20 of us] are all looking to get the same security back at the same time [...], 19 of us are gonna have a problem”
6.2 Exogenous Default and Propagation

In this model, default and fails are purely opportunistic and apply to securities independently from one another. A default on a security does not affect the remaining business of a trader or his ability to access spot markets. Precisely, an agent fails when the spot market price of the (unsegregated) tree exceeds the payment from the security he holds. The availability and liquidity of the spot market trade also facilitates rehypothecation. Indeed, it also allows to settle positions smoothly along a collateral re-use chain even when some agent defaults. The tree previously held by this agent will be available in the spot market to settle remaining trades.

In financial markets, default or fails may have more severe consequences or arise for exogenous motives. Banning defaulting agents from spot market would have little impact in this model. Indeed, Lemma 2 shows that we can consider default and fail-free securities without loss of generality. However, rehypothecation may amplify the impact of non-opportunistic default. Suppose indeed that an agent \( i \) may die between period 0 and period 1 with some exogenous probability \( \eta \). In this case, any unsegregated collateral held by agent \( i \) is lost. Creditors may keep the collateral they received from this agent while debtors can only collect the segregated fraction of the asset they posted. Without rehypothecation, all the collateral is segregated. Hence both creditors and debtors are better-off ex-post after agent \( i \)'s default. Importantly, they can still pay or deliver the asset for all transactions not implying agent \( i \)'s default. With rehypothecation, the tree available in the economy to settle positions is reduced by \( \sum_j (1 - \alpha_j) \bar{k}_j \phi_i^j \), the amount of unsegregated collateral held by agent \( i \). Hence, there might not be enough collateral to settle all remaining positions in the economy if (10) holds. Indeed, in this case, some of the unsegregated collateral held by agent \( i \) might be crucial to settle other positions. To fix ideas, consider against Example 2 of Section 5.2. If agent \( A \) disappears, there is only 3/4 units of collateral while the total demand from \( B \) for settlement with \( C \) amounts to 1 unit. Hence, a fraction 1/4 of type \( B \) agents must default. In addition, the price of the tree shoots up because of scarcity, affecting payoffs of type \( B \) agents who actually deliver. Hence, with rehypothecation, an exogenous default of type \( A \) agents would trigger endogenous default from some type \( B \) agents.

Both these examples show the importance of collateral scarcity in generating fragility.
during settlement. In particular, condition (10) highlights the importance of aggregate leverage in the economy. Indeed, when (10) holds, many credit transactions are ultimately backed by a limited amount of pledgeable asset. Aggregate leverage paves the way for liquidity issues in the asset market or contagion through credit chains. In this sense, our analysis suggests that any policy addressing rehypothecation should have a macroprudential focus.

7 Conclusion

This paper introduces rehypothecation in a competitive economy where agents post collateral to short securities. Re-use may facilitate the circulation of collateral but limited commitment also affects the implicit asset lending transaction. I show that rehypothecation can be replaced by an efficient financial structure without rehypothecation which delivers the same velocity of collateral. This result is robust to market segmentation whereby agents trade on decentralized markets with local clearing constraints. Hence, my results mitigate the claim from bankers and some academics that a ban on rehypothecation would severely affect secured financial markets. In the presence of market incompleteness however, the irrelevance result breaks down as rehypothecation allows to free up (inefficiently) encumbered collateral and increase welfare. Decentralized trading magnifies these gains as intermediaries typically need to take long and short positions simultaneously, which are costly in terms of collateral use. Finally, I showed that when aggregate collateral is scarce, settlement of positions along collateral chains is fragile.

This theory of rehypothecation fundamentally stresses the relationship between market incompleteness and collateral velocity. In general, other frictions not modeled in this paper could explain why dealer banks find it valuable to re-use collateral pledged by customers. The rough analysis of Section 6 also calls for a better understanding of collateral circulation and settlement of positions with re-used collateral.

Since rehypothecation does not improve upon our complete market benchmark, this analysis further stresses the importance of asset availability for collateralized financial markets. A clear consensus about the empirical relevance of such asset scarcity has yet

25Singh (2011) reports that collateral desk at dealer banks should be “self-funding” to avoid dipping in their own balance sheet for collateral. On a similar note, Kirk et al. (2014) believe that dealer banks might find it “uneconomical” to use their own collateral.
to emerge (cf CGFS (2013) for a recent overview). Still, the increasing demand for high quality assets to be used as collateral appears as a major trend in financial markets. In particular, the regulatory effort towards more central clearing of OTC trades may have significant effects on collateral demand. The implications of these trends for asset prices, public and private asset supply as well as unconventional monetary policy still rank high on research agendas.

References


CGFS (2013). Asset encumbrance, financial reform and the demand for collateral assets. Cgfs papers, BIS.


Appendices

A Sequential trading and settlement

In this section, I present a variation of the model with the same physical environment but where trading and settlement takes place over several subperiods. This departure allows to get a better understanding of collateral circulation and accommodates the need for sequential settlement when collateral is scarce and Assumption A does not hold.

I show that there exists a way to settle trades in an orderly fashion whereby securities traded first are settled last and conversely. In this configuration, by definition, collateral needs to deliver may not exceed the amount pledged during the corresponding trading round. In the absence of time discounting, agents agree to wait in line for settlement. In this context, in spite of its apparent importance, Assumption A is immaterial to our results in the following sense. Any equilibrium allocation of the model in the main text obtains as an equilibrium of the model introduced below.
Period 0 (resp. period 1) now contains $T$ trading (resp. settlement) rounds. In each trading round $\tau \in \{1, \ldots, T\}$, agent $i$ may buy a quantity $\theta_{i,0}^{B,\tau}$ of the tree and take positions $(\phi_{i,j}^{1,\tau}, \phi_{i,j}^{2,\tau}) \in \mathbb{R}^2$ for securities $j \in \mathcal{J}$. In round $\tau$, the tree trades competitively at price $p_{0,\tau}$ and security $j$ trades competitively at price $q_{i,\tau,j}$. In period 1, uncertainty is resolved and the settlement stage takes place in a backward fashion. Securities traded in trading round $\tau \in \{1, \ldots, T\}$ are settled\textsuperscript{26} in settlement round $T - \tau + 1$. The tree, which pays dividends after the last settlement round, is traded at price $p_{1,\tau}(s)$ in round $\tau$ and state $s$. Trading the tree after uncertainty is resolved allows agents to get back the collateral they hypothecated for delivery. This motive for trade also explains why the tree price might differ from the service flow, i.e. the dividend. Consumption in period 0 and 1 take place at the last round of the trading and settlement stages respectively. For every variable, subscript 0 (resp. 1) denotes the trading (resp. settlement) stage as before. The second subscript $\tau$ denotes the round. Figure 3 presents the time-line of the sequential trading model.

In the exposition, I implicitly conjectured that securities $j$ traded at different rounds $\tau$ and by different agents $i$ are homogeneous. Now, homogeneity might be an issue if the tree price changes during settlement as incentives to pay and deliver depend on the future price sequence of the tree. Second, besides the tree price, liquidity constraints will now affect traders’ ability to pay or deliver. The rest of this section should convince the reader that these concerns can eventually be dismissed.

A.1 Period 0: Trading stage

Agent $i$ enters trading round $1 < \tau \leq T$ with (i) a quantity $\omega_{i,0,\tau-1}$ of consumption good (also called cash here), (ii) a quantity of tree $\theta_{i,0,\tau-1}$ and (iii) a record of securities traded during the $\tau - 1$ previous rounds $\Phi_{i-1}^\tau = (\phi_{i,1}^{1,\tau-1}, \ldots, \phi_{i,\tau-1}^{1,\tau-1}) \in \mathbb{R}^{2J \times (\tau - 1)}$. In line with the main model, the initial values of the variables introduced above are $\omega_{i,0}^0 := \omega_{i,0}^0$ and $\theta_{i,0}^0 := \theta_{i,0}^0$. During round $\tau$, given respective prices $(p_{0,\tau}, q_{\tau})$, he chooses new purchases of collateral $\theta_{i,0}^{B,\tau}$ and securities $\phi_{i}^\tau \in \mathbb{R}^{2J}$. The following equations then describe the evolution of the

\textsuperscript{26}Although it simplifies the exposition, this feature is not crucial for the result. In particular, agents may be allowed to settle before $T - \tau + 1$ or at any round of the settlement stage
Trading Round \( (\omega_{\tau-1}^i, \theta_{\tau-1}^i, \Phi_{1\tau-1}) \)

Consumption 
\[ t = 0 \]

\[ \text{Settlement Claims} \]
\[ T - \tau + 1 \]

\[ \text{Consumption} \]
\[ t = 1 \]

Trading Stage

Settlement Stage

Figure 3: Timeline of the sequential trading model

Naturally, the agent consumes in period 0 whichever amount of consumption good he has left after he conducted his trades, that is

\[ c_i^0 := \omega_{0,T}^i \]  

(13)

Furthermore, cash and collateral account balances must verify the following constraints in any period \( \tau \)

\[ \theta_{0,\tau-1}^i + \theta_{0,\tau}^i \geq \sum_j \tilde{k}_j \phi_{r,j}^{i,-} \]  

(14)

\[ \omega_{0,\tau}^i \geq 0 \]  

(15)

Inequality (14) is the collateral constraint for trading round \( \tau \). Its formulation explains the main difference with the sequential model of the main text. To account for

\footnote{When he takes long positions in securities, agent \( i \) also adds up \( \sum_j \alpha_j k_j \phi_{r,j}^{i,+} \) to his segregated account. For expository convenience, I abstract from the mechanical accounting of tree inflows in the segregated account}
collateral circulation in a consistent way, tree received from long positions may only be used (pledged or sold) one trading round after it was acquired. Constraint (15) requires agents to hold positive cash holdings at the end of every trading round. Sequential trading differs from simultaneous trading of the main text because these two liquidity constraints may bind. Most importantly a buyer must put the security price up front, as the non-segregated tree can only be sold in the next trading round.

A.2 Period 1: Settlement stage

In settlement round $1 \leq \tau \leq T$, only security traded at round $T - \tau + 1$ are settled if any. A trader $i$ with a short position in security $j \phi^i_{j,T-\tau+1} > 0$ either repays the face value $\bar{R}_j(s)$ or defaults. A trader $h$ with a long position $\phi^h_{j,T-\tau+1} > 0$ decides either to turn back the unsegregated collateral $(1 - \alpha_j)\bar{k}_j$ or fails. Since uncertainty is resolved before settlement takes place, I abstract from indexing variables by the realized state $s$ in the following.

As in the trading stage, I introduce the variables summarizing an agent’s position upon entering settlement round $\tau$, i.e. (i) consumption good (sometimes referred to as cash) in quantity $\omega^i_{1,\tau-1}$, (ii) a quantity of tree $\theta^i_{1,\tau-1}$ and (iii) securities yet to be settled $\Phi^i_{T-\tau+1} = (\phi^i_{1,\tau}, ..., \phi^i_{T-\tau-1}) \in \mathbb{R}^{J \times (T-\tau-1)}$. At every settlement stage $\tau$, an agent must decide whether to default on securities he shorted at trading stage $T - \tau + 1$ ($d^i_{\tau \in \{0,1\}}$) and whether to deliver collateral on his long positions ($f^i_{\tau \in \{0,1\}}$). To deliver collateral, he may either use his tree holdings or buy tree (quantity $\theta^{i,B}_{1,\tau}$) in the collateral market of round $\tau$.

The initial values of the variables introduced above are $\omega^i_{1,0} := \omega^i_1$ and $\theta^i_{1,0} := \theta^i_{0,T}$. In each settlement round $\tau$, given the tree price sequence $(p_{1,\tau}, ..., p_{1,T})$ agent $i$ chooses optimally the tree purchase $\theta^{i,B}_{1,\tau}$, the default $d^i_{\tau}$ and fail $f^i_{\tau}$ in order to maximize his period 1 consumption

$$c^i_{1} = \omega^i_{1,\tau} + \theta^{i}_{1,\tau}x$$  \hspace{1cm} (16)

In every round $\tau$, payments and deliveries follow the same Default Resolution Mechanism as in the main model. First, sellers may default by not paying the face value in which case they lose the collateral attached to the loan. Second buyers may fail by

\hspace{1cm} 28Again, to streamline the presentation, I abstract from describing the movements of tree in the segregated account.
keeping the unsegregated collateral in which case they do not receive the face value of the claim they purchased. Finally, actual payments and deliveries are ultimately conditional on counterparties’ decisions which are taken as given. During the settlement stage, the cash and collateral accounts of agent $i$ evolve as follows:

$$\omega_{i,\tau} = \omega_{i,\tau-1} - p_{i,\tau} \theta_{i,\tau} + \sum_j (1 - f^i_{j,\tau})(1 - d_{j,\tau}) \bar{R}_j \phi_{T - \tau + 1, j}^{b^+} - \sum_j (1 - d^i_{j,\tau})(1 - f_{j,\tau}) \bar{R}_j \phi_{T - \tau + 1, j}^{b^-}$$

$$\theta_{i,\tau} = \theta_{i,\tau-1} + \theta_{i,\tau}^{B} - \sum_j (1 - f^i_{j,\tau})(1 - d_{j,\tau})(1 - \alpha_j) - d_{\tau,j}\alpha_j \bar{k}_j \phi_{T - \tau + 1, j}^{b^+}$$

$$+ \sum_j (1 - d^i_{\tau,j})\bar{k}_j (1 - f_{\tau,j}) + \alpha_j f_{\tau,j} \phi_{T - \tau + 1, j}^{b^-}$$

Equation (17) rules out inter-round credit because of limited commitment. Equation (20) states that a given piece of collateral can only be used to settle one transaction at a time. These trading constraints may generate frictions along the sequential delivery process. Indeed, collateral received in one round cannot be redeployed (sold or delivered) in the very same round: hence borrowers must put up front the face value of the corresponding security $^29$

Let us call $d^{-i}$ and $f^{-i}$ the $T \times (J - 1) \times (I - 1)$ vectors collecting decisions to fail and default by agents other than $i$. We label $c^i_1(\omega^i_1, \theta^i_{0,T}, \Phi^i_{1T}, p_1, d^{-i}, f^{-i})$ the optimal choice of a consumer with initial balances $(\omega^i_1, \theta^i_{0,T}, \Phi^i_{1T})$, taking as given the sequence of collateral price $p_1$ and other traders’ decisions $(d^{-i}, f^{-i})$ where trader $i$ optimizes with respect to $(\theta^i_{1,T}, d^i_{\tau,j}, f^i_{\tau,j})_{\tau=1..T}$ given constraint (19)-(20).

$^29$We could suppose in addition that a lender cannot pay for the tree attached to a security with the cash obtained from the borrower. This additional constraint would require to tighten inequality (19) into:

$$\omega_{i,\tau-1} - p_{i,\tau} \theta_{i,\tau}^d - \sum_j (1 - d^i_{j,\tau})(1 - f_{j,\tau}) \bar{R}_j \phi_{T - \tau + 1, j}^{b^-} \geq 0$$

(19b)
Definition 5: Given initial positions \((\omega^i_1, \theta^i_{0,T}, \Phi^i_{1T})_{i \in \mathcal{I}}\), and a state \(s \in \mathcal{S}\), a settlement equilibrium is a price sequence \(p_1 = (p_{1,1}, \ldots, p_{1,T})\), payments and delivery decisions \(\{d^i_\tau, f^i_\tau\}_{\tau=1,T}\) for every agent \(i \in \mathcal{I}\) and an allocation \(\{c^i_1\}_{i \in \mathcal{I}}\) such that

1. \(p_{1,\tau}\) clear the collateral market in round \(\tau\)
2. \(\forall i \in \mathcal{I}, c^i_1 = c^i_1(\omega^i_1, \theta^i_{0,T}, \Phi^i_{1T}, p_1, d^{-i}, f^{-i})\).
3. \(\{d^i_\tau, f^i_\tau\}_{\tau=1,T}\) is consistent with individual decisions to default and fail.

The main variable of interest in a settlement equilibrium is the price sequence for the tree \(\{p_{1,\tau}\}_{\tau=1,T}\) which determines default and fail decisions. The following proposition shows that with an orderly sequential settlement process, these prices are generically equal to the dividend paid by the tree.

Proposition 3 Let initial positions \((\omega^i_1, \theta^i_{0,T}, \Phi^i_{1T})_{i \in \mathcal{I}}\), and a state \(s \in \mathcal{S}\) be given.

1. There exists a fundamental settlement equilibrium with a price sequence \(p_{1,\tau} = x\) for all \(\tau\).
2. If

\[
\forall \tau \in \{1, \ldots, T\}, \quad \sum_{i \in \mathcal{I}} \theta^i_{0,\tau-1} > \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \bar{k}_j \phi^{i,j}_{\tau,j} \tag{21}
\]

holds, this is the only equilibrium.

First, let us explicit condition (21). By construction, in any trading round, the amount pledged can be no greater that the quantity of non-segregated tree in the economy inherited from last period. This writes

\[
\forall \tau \in \{1, \ldots, T\}, \quad \sum_{i \in \mathcal{I}} \theta^i_{0,\tau-1} \geq \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{J}} \bar{k}_j \phi^{i,j}_{\tau,j}
\]

Hence, condition (21) fails to hold when all the tree is pledged as collateral in some round \(\tau\).

Proof ⊳ Since incentives to default and fail depend on the whole future sequence of prices, we use backward intuition. Consider the last round of settlement where claims
traded first are settled. Collateral demand to deliver is bounded above by \( \sum_i \sum_j (1 - \alpha_j) \bar{k}_j \phi_{1,j}^- \). This is because some lenders might consider failing rather than delivering. Obviously, this amount is lower than the quantity of non-segregated collateral after round 1 trades. This writes

\[
\sum_i \sum_j (1 - \alpha_j) \bar{k}_j \phi_{1,j}^- \leq \theta_0 - \sum_i \sum_j \alpha_j \bar{k}_j \phi_{1,j}^-. \tag{22}
\]

From the settlement point of view, the RHS is also the available quantity of tree before this last round of settlement takes place. Since the consumption value of the tree is equal to \( x \), demand for the tree at any price \( p_{1,T} > x \) is bounded above by the LHS while supply at such a price is bounded below by the RHS. It is then clear that \( p_{1,T} \) is an equilibrium price for the last settlement round and that this is the only equilibrium price if (22) is an inequality.

We can now iterate the argument backward. Consider settlement round \( \tau < T \) given future price sequence \( p_{1,\tau'} = x \) for all \( \tau' > \tau \leq T \). Again, in round \( \tau \), the demand of collateral to deliver is bounded above by the supply. The same argument allows to prove that \( p_{1,\tau} = x \) is an equilibrium price in round \( \tau \) and that this is the only one if

\[
\sum_{i \in I} \bar{\theta}_{i,\tau-1} < \sum_{i \in I} \sum_{j \in J} \bar{k}_j \phi_{\tau,j}^- \quad <
\]

As we observed, if (21) does not hold, the tree may trade at a price superior to the fundamental value \( x \) during the trading stage. This pattern emerges because lenders may attach a high marginal value to the tree. When the non-segregated tree is scarce, this value to deliver drives up the price of the tree. To better understand this phenomenon, consider again Example 1.

There is only one settlement round (although there are formally two trading rounds). In order to return collateral to agent \( C \), agent \( B \) needs to buy back 1/2 units of tree from agent \( A \). Consider state \( s = 1 \) and let \( p_1(1) \) be the price of the tree. The net gain from delivering is \( 1 - 1/2p_1(1) \) which is the bond’s payoff minus the cost of retrieving the re-used collateral. Failing yields 0. This explains why agent \( B \) would pay up to \( \bar{p}_1(1) = 2 = 2 \times x(1) \) for the tree. In aggregate, agent \( B \) needs 1/2 units which is precisely the supply of non-segregated tree. Hence the value to deliver determines the price and
any value $p_1(1) \in [1, 2]$ can support the settlement equilibrium. Although interesting, this effect is not robust. Indeed, it would vanish with any infinitesimal reduction in agent $B$ sale to agent $A$.

Equipped with the settlement equilibrium concept, we can naturally define an equilibrium of the sequential model.

**Definition 6: Equilibrium** An equilibrium of the sequential model is a trading price vector $(p_0, q_0) \in \mathbb{R}^{(J+1) \times T}$, a settlement price vector $(p_1(1), ..., p_1(S)) \in \mathbb{R}^{S \times T}$, a default and fail map $(d, f) \in \mathbb{R}^{J \times I \times S \times T}$ and an allocation $(c_0, c_1) \in \mathbb{R}^{I \times (S+1)}$ such that

1. $\forall s \in S, (c_1(s), p_1(s), d(s), f(s))$ form a settlement equilibrium given $(\omega_i^0(s), \theta_i^0, \Phi_i^0)_{i \in I}$

2. $\forall i \in I$,

$$
(\theta_i^0, \Phi_i) \in \arg \max (\omega_i^0) + S \sum_{s=1}^{S} \pi(s)u^i(c_1(s))
$$

where $\omega_i^0$ is defined by (11) − (15)

3. Agents form correct expectations about $p_1, d$ and $f$.

The sequential equilibrium concept is significantly more involved than that of the main text. First, since trade and delivery take place sequentially, agents face liquidity constraints. It may prevent a simultaneous trade pattern to obtain as a sequence of transactions.

### A.3 Equivalence between the models

In this section, we show that any equilibrium of the simultaneous ($Sim$) model can be supported as an equilibrium of the sequential ($Seq$) model. The liquidity constraints in the sequential model are twofold. During the trading stage, a buyer cannot partially finance a security purchase by selling the non segregated tree in the same round. During the settlement stage, an agent must put the face value of a security he shorted up front, except for the last settlement round (that is, he cannot sell the collateral returned to him in the same round). The next proposition shows that these frictions may be overcome.
**Proposition 3**

Every equilibrium allocation of model Sim is an equilibrium allocation of model Seq.

**Proof**

\(\triangleright\) Let \((c, p, q)\) be an equilibrium of the Sim model. By definition, for every trader \(i\), there exists a portfolio \((\theta^i, \phi^i) \in \mathbb{R} \times \mathbb{R}^L_+\) which finances \(c^i\) under \((p, q)\). We want to show that \(c\) can be obtained as an equilibrium of the Seq model.

For this, we obviously focus on a fundamental settlement equilibrium of the Seq model with \(p_1, \tau = x\) for all \(\tau\). In addition, the relevant trading prices are \(p_0, \tau = p\) and \(q_\tau = q\) for any \(\tau\), i.e. constant collateral and security prices. Denote \(SimBC(p, q, \omega_0, \theta_0)\) (resp. \(SeqBC_T(p, q, \omega_0, \theta_0)\)) the budget constraint in the Sim model (resp. the Seq model with \(T\) trading rounds). For the sequential budget constraint, remember that the first two arguments are formally sequences and that we omitted the price sequence for the tree at the settlement stage. The result obtains if budget feasible allocations are the same in the following sense.

i) For every \(T\), \(SeqBC_T(p, q, \omega_0, \theta_0) \subset SimBC(p, q, \omega_0, \theta_0)\)

ii) \(SimBC(p, q, \omega_0, \theta_0) \subset SeqBC_T(p, q, \omega_0, \theta_0)\) for some \(T\).

Point i) is easy to show as an agent faces the same prices but additional constraints with the sequential budget set, namely, liquidity constraint (15) and collateral constraint (14).

Point ii) is slightly more involved because of the liquidity constraints. For a given agent (where we omit the superscript \(i\)), let us consider the following trades over two rounds:

1. Buy \(\theta^B_{0,1} = \theta - \theta_0 + \frac{1}{p}q_\tau \phi^+\) units of tree and short \(\phi^-\)

2. Buy \(\theta^B_{0,2} = -q_\tau \phi^+\) units of tree and buy \(\phi^-\)

It is easy to see that as a result of the two operations, the agent effectively holds total portfolio \((\theta, \phi)\) which finances \(c\) under prices \((p, q)\). We are left to verify that liquidity constraints in trading round 1 and settlement rounds 2 are verified.

For the first point, use the fact that \((p, q)\) are equilibrium prices of the simultaneous
model and hence verify absence of arbitrage. In particular, AOA equation (23) implies that
\[
\left(\frac{1}{p}q - (1 - \alpha)\bar{k}\right)\phi^+ \geq 0
\]
(23)
Thus, for the collateral constraint observe that
\[
\theta^U_{0,0} + \theta^B_{0,1} = \theta_0 + \theta_0 + \frac{1}{p}q\phi^+ \geq \theta + (1 - \alpha)\bar{k}.\phi^+ \geq \bar{k}.\phi^- = \bar{k}.\phi^-_1
\]
where the first inequality uses (23), the second is the collateral constraint (6) of the simultaneous model.

As for the liquidity constraint (15), note that
\[
\omega_{0,1} = \omega_0 - p\theta^B_{0,1} + q\phi^- = \omega_0 - p(\theta - \theta_0) - q\phi^+ + q\phi^- = c_0 \geq 0
\]
where the inequality follows from \(c \in SimBC(p, q, \omega_0, \theta_0)\)

Consider now settlement round 2. Since this is the last settlement round, the agent may cash in the tree dividend to pay for short positions traded in round 1. Hence the cash in advance constraint for settlement does not bind because all the short positions where traded in trading round 1.

Finally, since budget feasible plans are the same and \(c\) is optimal in the simultaneous model under prices \((p, q)\), this is also the consumer’s choice in the sequential model under \((p, q)\). \(\triangleright\)

**B Heterogeneous Valuation for the Durable Asset**

Abstracting from its use as collateral, the valuation for the tree derives from the dividend \(x(s)\) paid in the single perishable consumption good in period 1. With a single con-
sumption good, agents may not value differently this service flow from the tree\(^{31}\). In this section, I discuss the role of rehypothecation when allowing for such heterogeneity. This new environment highlights the relationship between the desirability of rehypothecation and spot market liquidity.

Suppose there are now two consumption goods in the economy. Good 1 is the perishable good of the main model, used as numeraire in this environment. The tree pays dividends \(x\) in perishable good 2 which is not a perfect substitute for good 1. In period \(t = 0, 1\), \(c_{ik}(s)\) denotes agent \(i\) consumption of good \(k = 1, 2\) in state \(s \in S\). Within periods references over the two goods and represented by the instantaneous utility function \(v^i(c_1, c_2)\).

For the sake of simplicity, I further develop the discussion within a variant of Example 2 from Section 5.2. There are 3 agents \(A, B, C\) and 2 states of the world \(s = 1, 2\) with probability \(\pi(s) = 1/2\). Trading is bilateral because \(A\) and \(C\) can only trade with \(B\). The tree pays off \(x(s) = s\) units of good 2 for \(s = 1, 2\) in period 1 and still no dividend in period 0. Instantaneous preferences over the two goods are given by

\[
v^i(c_1, c_2) = u(c_1 + r^i c_2) \quad \text{with} \quad r^A = r^B = 1, \quad r^C \geq 1
\]

where \(u\) is \(C^2\), strictly monotone and concave and satisfies the Inada condition: \(\lim_{c \to 0} u(c) = +\infty\). When \(r^C > 1\), agent \(C\) values the service flows of the collateral strictly more than agents \(A\) and \(B\). Hence, any ex-post efficient allocation has agent 3 holding the tree entirely. Let \(e \in (2r^c + 1/2, \infty)\) be given and consider the following endowment pattern:

\[
\begin{align*}
\omega_{01}^A &= e + 1/2 \\
\omega_{11}^A(s) &= e - 1/2 \\
\theta_0^A &= 0 \\
\omega_{01}^B &= e + 1/2 \\
\omega_{11}^B(s) &= e - 1/2 \\
\theta_0^B &= 0 \\
\omega_{01}^C &= e - 1 \\
\omega_{11}^C(s) &= e + 1 - r^C s \\
\theta_0^C &= 1
\end{align*}
\]

The lower bound on \(e\) implies that agent \(C\) will always be the marginal holder of the tree if a spot market is available at date 1. The tree dividend in period 1 is the only source of good 2 in the economy, hence the omission above. Finally, agents can trade a non-contingent bond which promises 1 units of good 1 in any state \(s = 1, 2\), collateralized by

\(^{31}\)In a mortgage loan for instance, the house owner derives a higher utility from the house than a bank that would repossess it. In financial markets, financial institutions might face different regulatory constraints and hence derive utility from holding safe or liquid assets.
one unit of the tree\textsuperscript{32}. Observe that the environment is exactly that of Section 5.2 when $r^C = 1$. When $r^C > 1$, the efficient symmetric allocation $c^*$ is the following

$$\text{For } i = A, B, \quad (c_{i1}^*, c_{i2}^*) = (e, 0) \quad (c_{01}^*, c_{02}^*, c_{11}^*(s), c_{12}^*(s)) = (e, 0, e - r^C s, 1)$$

Hence, every agent consumes $e$ units of the composite consumption good, exactly as before. However, it is now crucial for efficiency that agent $C$ be the final holder of the tree because he values strictly more its fruits.

\textit{Rehypothecation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure4.png}
\caption{Financing $c^*$ with re-use}
\end{figure}

Rehypothecation allows to decentralize this allocation with agent $C$ retaining ownership of the tree. In fact, trades described in Section 5.2 and illustrated in Figure 4, still finance the optimal allocation $c^*$. Agent $C$ pledges the tree to agent $B$ to borrow 1. Agent $B$ re-uses half of the tree (the other half being segregated) to borrow $1/2$. The tree is then returned to agent $C$ when trades are settled. Indeed, observe that agent $i$ repays a loan if

$$r^i x(s) - 1 \geq 0$$

and delivers the collateral if

$$1 - 1/2 r^i x(s) \geq 0$$

Since $r^A = r^B = 1$, agents $A$ and $B$ option value of default or failure remain negative as in the main text. Agent $C$ incentives to repay are actually strengthened since $r^C \geq 1$.

\footnote{Intuitively, with heterogenous valuation for the tree, collateral requirements could dependent on type as $C$’s borrowing capacity increases with $r^C$. However this is not necessary to reach the efficient allocation with rehypothecation}
In the incentives constraints above, the term $r^i x(s)$ reflects the private value of one unit of tree to agent $i$ but not his market price which would be $r^C x(s)$ with our assumption on $e$. However, our argument with rehypothecation precisely obtains without a spot market for the tree in period 1. Importantly, rehypothecation allows agent $C$ to retain ownership of the tree while giving up possession for $B$ to use as collateral\textsuperscript{33}.

\textit{No Rehypothecation}

Collateral re-use is now prohibited but agents may trade any state contingent security. It is immediate to see that the efficient allocation $c^*$ can only be implemented if agents trade the tree in period 0. Suppose not. Observe that agent $A$ needs to receive a positive transfer $(1/2, 1/2)$ of good 1 to finance $c^A*$ which requires agent $B$ - his sole counterparty in this economy - to take the corresponding short position. This is feasible only if $B$ holds some of the tree to use as collateral, a contradiction.

Hence, without a spot market for the tree in period 1, there exists a trade-off between consumption smoothing and ex-post efficiency. Because $B$ needs to borrow from $A$, he must hold the tree as collateral. However $C$ should hold the tree since he values it strictly more than $B$. This argument has no bite in the main model as the second channel is switched off with a single consumption good.

However, it may seem natural to open a spot market to trade good 1 and 2 in period 1. The (good 1) price of the asset (or equivalently the rental price of the service flow) would become $p_1(s) = r^C x(s)$ to reflect trader $C$ higher valuation for the tree. Let us now write the payoff matrix (in numeraire good 1) for the asset and two extremal securities $j_1$ and $j_2$ (requiring 1 and 1/2 units of tree as collateral).

\[
R(r^C) = \begin{bmatrix}
  r^C & r^C & 0 \\
  2r^C & 0 & r^C
\end{bmatrix}
\]

Then it is easy to check that the following portfolios

\[
(\theta^A, \phi^A_{j_1}, \phi^A_{j_2}) = \left(\frac{1}{2r^C}, 0, -\frac{1}{2r^C}\right) \quad (\theta^B, \phi^B_{j_1}, \phi^B_{j_2}) = \left(0, \frac{1}{2r^C}, \frac{1}{2r^C}\right) \quad (\theta^C, \phi^C_{j_1}, \phi^C_{j_2}) = \left(1-\frac{1}{2r^C}, -\frac{1}{2r^C}, 0\right)
\]

implement the desired allocation. With respect to the standard case $r^C = 1$, the portolios

\textsuperscript{33}See Singh (2014) for an interesting discussion of this point regarding recent unconventional monetary policy operations by the Federal Reserve
have been scaled down by $r^C > 1$ to reflect the higher collateral value of the tree resulting from agent $C$’s higher valuation.

While the general analysis is beyond the scope of this paper, I conjecture that rehypothecation would not yield strict welfare gains in this context. This should not come as a surprise in the light of Proposition 2. Indeed, the irrelevance result states that strict welfare gains from collateral re-use may only materialize in an economy with incomplete markets. With heterogeneous valuation for the asset, market completeness also requires a spot market for the tree in period 1.

C Proofs

C.1 Proof of Lemma 1

\[ \Delta \]

Let $E := (c, p, q)$ be an equilibrium in economy $\mathcal{E}(\mathcal{J} \cup \mathcal{J}_1)$ where $\mathcal{J}_1 \in Sp(\mathcal{J})$. By definition, for every agent $i$, there exists a portfolio $(\theta^i, \phi^{i+}, \phi^{i-}) \in \mathbb{R} \times \mathbb{R}^{\frac{\mathcal{J} \cup \mathcal{J}_1}{\phi}} \times \mathbb{R}^{\frac{\mathcal{J} \cup \mathcal{J}_1}{\phi}}$ of collateral and securities of $\mathcal{J} \cup \mathcal{J}_1$ which finances the allocation $c^i$ under prices $(p_0, q_0)$ according to budget constraint (4)-(6). In what follows, I show that $c$ is an equilibrium allocation of economy $\mathcal{E}(\mathcal{J}_1)$ where only securities in $\mathcal{J}_1$ are available for trade. Let $E_1 = (c, p_1, q_1)$ this equilibrium to be constructed.

Every security is priced in $E$, even if it is not traded. Let us set prices for securities $(p_1, q_1)$ of $\mathcal{J}_1$ in $E_1$ to their value in $E$. Since $\mathcal{J}_1 \subset \mathcal{J}$ budget feasible allocations in $E_1$ are feasible in $E$.

Suppose now a security $j = (\bar{R}_j, \bar{k}_j, \alpha_j) \in \mathcal{J} \setminus \mathcal{J}_1$ is traded in equilibrium $E$. By definition of $\mathcal{J}_1$, there exists a replicating portfolio $\psi(j) = (\theta(j), \phi(j))$ verifying criteria (i) – (iii) of Definition 3. For any agent $i$, replace every long position $\phi^{i+}_j$ (resp. short position $\phi^{i-}_j$) in security $j$ by $\phi^{i+}_j$ (resp. $-\phi^{i+}_j$) units of portfolio $\psi(j)$. The following points prove that the substitution achieves our goal

a) Using market clearing for security $j$ in $E$, the securities in $\psi(j)$ verify market clearing in $E_1$. To put it otherwise, the substitution is resource neutral.

b) Second, any agent’s payoff in period 1 stays identical by definition of $\psi(j)$. Furthermore, the replicating portfolio’s price in $E_0$ equals that of the security. If the former were strictly lower, by monotonicity of preferences long agents would have
bought $\psi(j)$ instead of $j$. If it were strictly higher, short agents would have sold $\psi(j)$ instead of $j$. Hence, the substitution is also cost neutral.

c) Finally, substituting $\psi(j)$ for $j$ does not violate the collateral constraint. By construction the substitution does not require more collateral or more segregation. To see this, consider a long agent first. The net variation in the collateral constraint from this substitution is

$$\Delta \theta^+ = -(1 - \alpha_j)\bar{k}_j + \theta(j) + \sum_{j_1 \in \psi(j)} (1 - \alpha_{j_1})\phi_{j_1}\bar{k}_{j_1} \geq 0$$

The term to enter negatively is the quantity of collateral that can be repledged out of 1 unit of security $j$. The positive terms are the quantity of tree in the replicating portfolio and the re-usable collateral in the securities of $\psi(j)$.

For an agent short in $j$, the substitution yields

$$\Delta \theta^- = +k_j - \theta(j) - \sum_{j_1 \in \psi(j)} \phi_{j_1}\bar{k}_{j_1} \geq 0$$

The term to enter positively is the collateral requirement for $j$. The first negative term accounts for the sale of $\theta(j)$ units of tree while the second one represents the collateral requirement to short the securities in $\psi(j)$.

Hence, we have shown that $c$ is budget feasible with securities in $J_1$. Since this is the optimal choice of agents under a larger budget set, $c$ is the optimal choice of agents in $E(J_1)$ and thus constitutes an equilibrium allocation. $\triangleright$

### C.2 Proof of Lemma 2

Let $j = (\bar{R}_j, \bar{k}_j, \alpha_j) \in J_0$ and consider security $j' = (R_j/\bar{k}_j, 1, \alpha_j)$ the face value of which is proportional to the actual payoff of $j$. Since $\bar{R}'_j(s) \in [(1 - \alpha_j)x(s), x(s)]$, we have $R'_j(s) = \bar{R}'_j(s) = (1/\bar{k}_j)R_j$. Hence, security $j$ can be replicated by $\bar{k}_j$ units of security $j'$. It is thus enough to find a replicating portfolio for $j'$. We can then restrict our attention to the following set :

$$J_1 = \left\{ (\bar{R}_j, \bar{k}_j, \alpha_j) \mid \bar{R}_j(s) \in [(1 - \alpha_j)x(s), x(s)], \bar{k}_j = 1, \alpha_j \in [0, 1] \right\} \subset J_0$$

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which is exactly the set of no-default/no-fail securities collateralized by one unit of the tree. Let now \( j = (\bar{R}, 1, \alpha) \in \mathcal{J}_1 \) and re-order the states \( s = 1, .., S \) so that

\[
1 \geq \frac{\bar{R}(1)}{x(1)} \geq \frac{\bar{R}(2)}{x(2)} \geq \cdots \geq \frac{\bar{R}(S)}{x(S)} \geq (1 - \alpha)
\] (24)

Whenever \( \bar{R}(s) = \bar{R}(s') \), let the initial ordering prevail. Next consider the \( S - 1 \) securities \( \{j_1, j_2, .., j_{S-1}\} := \mathcal{J}_*(j) \subset \mathcal{J}_* \) which verify

\[
\begin{align*}
R_{j_l}(s) &= x(s), \quad \text{if } 1 \leq s \leq S - l \\
R_{j_l}(s) &= 0, \quad \text{otherwise}
\end{align*}
\]

Security \( j_l \) has the same payoff as one unit of collateral in the first \( S - l \) state. I now derive the portfolio \( \psi(j) = (\theta, \phi_{j_1}, .., \phi_{j_{S-1}}) \in \mathbb{R}^{S}_+ \) of tree and securities of \( \mathcal{J}_*(j) \) to replicate \( j \). To this effect, set

\[
\begin{align*}
\forall l \geq 1, \quad \phi_{j_l} &= \frac{R(S - l)}{x(S - l)} - \frac{R(S - l + 1)}{x(S - l + 1)} \\
\theta &= \frac{R(S)}{x(S)}
\end{align*}
\]

By construction, this portfolio replicates security \( j \)'s payoff so that requirement (i) of Definition 3 holds. The collateral needed to sell portfolio \( \psi(j) \) is

\[
k(\psi(j)) = \theta + \sum_{l=1}^{S-1} \phi_{j_l} = \frac{R(1)}{x(1)} \leq 1 = \bar{k}_j
\]

Finally, the collateral segregated when selling \( \psi(j) \) is:

\[
k(\psi(j)) = \sum_{l=1}^{S-1} \alpha_{j_l} \phi_{j_l} = \sum_{l=1}^{S-1} \phi_{j_l} \frac{R(1)}{x(1)} - \frac{R(S)}{x(S)} \leq \alpha_j
\]

Hence conditions (ii) and (iii) also hold. We thus proved that any contract in \( \mathcal{J}_0 \) can be replicated by contracts in \( \mathcal{J}_* \), according to Definition 3. \(<\)

END