Sequential bidding in asymmetric first price auctions

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Abstract

We study asymmetric first price auctions in which bidders place their bids sequentially, one after the other and only once. We show that with a strong bidder and a weak bidder (in terms of first order stochastic dominance of their valuations' distribution functions), already with small asymmetry between the bidders, the expected revenue in the sequential bidding first price auction (when the strong bidder bids first) is higher than in the simultaneous bidding first price auction. Moreover it is higher than the expected revenue in the second price auction. The expected payoff of the weak bidder is also higher in the sequential first price auction. Therefore a seller interested in increasing revenue facing asymmetric bidders may find it beneficial to order them and let them bid sequentially instead of simultaneously. In terms of efficiency, both the simultaneous first price auction and the sequential first price auction cannot guarantee full efficiency (as opposed to a second price auction). The sequential bidding auction when the stronger bidder bids first achieves lower efficiency than the simultaneous auction. However, when the order is reversed and the asymmetry is large enough the sequential first price auction achieves higher efficiency than the simultaneous auction.

Keywords: Sequential bidding, asymmetric auctions, first price auction, second price auction.

JEL classification: D44

1 Introduction

The question of designing an auction mechanism that will ensure high revenue for the seller when bidders have private information on their valuation of the object and are ex ante asymmetric, has long been an open question. Even the more restricted question of finding equilibrium behavior and then ranking different known auctions according to their expected revenue is still unresolved in the general asymmetric case. We cannot even determine under what conditions will a first price auction (FPA) yield a higher expected revenue than

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a second price auction (SPA) when bidders are asymmetric.[1] Partial results exist for different assumptions on the distribution of bidders values.

Here we propose a simple modification of the first price auction in which bidders bid sequentially and only once. We order the bidders according to some order and bidder i observes the bids of previous bidders $1, ..., i-1$ when she places her own bid. As in the standard first price auction, the winner is the bidder who submitted the highest bid and she pays her bid. All other bidders have a payoff of zero. We break ties in favor of later bidders. Thus bidder i wins if her bid is higher than or equal to the bids of bidders $1, .., i-1$ and strictly higher than the bids of bidders $i+1, ..., n$. For the two bidders case, even with small asymmetry between bidders, we show that by ordering the bidders such that the strong bidder bids first and the weak bidder bids only after the strong bidder does[2], we increase both the weak bidder's expected payoff and the expected revenue compared to the simultaneous auction.

Asymmetry between bidders is common in many auctions. For example when one of the bidders is an incumbent firm while the other is an entrant firm it is plausible to assume asymmetry in their valuations for the object (which can be a result of differences in the cost function). Another source for asymmetry may be the seller's preferences over the bidders. In procurement auctions for example it is common that one seller is given an advantage over the other, reflecting better reliability or quality. Legal interventions that are intended for encouraging the participation of the weaker firm or preferred seller, such as *bidding credits* and *set-asides* are prevalent. Our results indicate that already with a small asymmetry the weaker firm should be given the opportunity to bid after the stronger firm bids. This not only improves the expected payoff of the weaker firm and therefore may increase participation of weaker firms, but also increases the expected revenue for the seller. In most common auctions bidders bid simultaneously or equivalently without knowing the bids of their opponents. However, in several settings the seller may approach the bidders one by one creating a sequential bidding game if she discloses the bids of previous bidders. Thus our motivation for studying sequential bidding FPA does not stem from an existing practice (we are not aware of such a practice) but from the theoretical understanding that altering the common practice to the sequential one may improve its performance.

Remarkably, the dynamic nature of bidding in the sequential bidding auction makes the game more tractable and as a result makes it simple to find the perfect Bayesian equilibrium for general distributions and a general number of bidders. This is yet another advantage of the sequential FPA over the simultaneous one where it is extremely difficult to find equilibrium behavior and analyze changes in the mechanism (such as handicaps or headstarts) on the bidders' behavior. We therefore suggest that sequential mechanisms be used when bidders are asymmetric to allow for higher revenue, higher participation of weaker bidders and easier prediction of expected behavior. Note moreover that even if the seller does not know the distributions

[1] Examples exist for both directions - a first price auction may either yield a higher or a lower expected revenue than a second price auction but no sufficient conditions are yet known for either of the directions.

[2] Formally we say that bidder i is stronger than bidder j if the distribution of her valuation, F_i, first order stochastically dominates the distribution of bidder j's valuation - F_j.

of the bidders' valuations but only that they are asymmetric such that one is strong and the other is weak our results suggest that by ordering them such that the strong bidder bids first he will almost always ensure a higher revenue than by a simultaneous auction.

We also examine the efficiency of the sequential first price auction. Efficiency is defined as the probability that the higher valuation bidder wins the object. When the bidders are asymmetric simultaneous first price auction does not guarantee efficiency. This happens because in equilibrium the weaker bidder bids more aggressively than the stronger one (does less shading down of her valuation) and therefore may win even if her valuation is lower than that of the stronger bidder. This is true also for the sequential first price auction. In the sequential auction the second bidder has an exogenous advantage of observing the first bid and being able to win by placing the same bid as the first bidder. Therefore, the second bidder may win although she has a lower valuation than the first one. We show that when the stronger bidder bids first the efficiency is lower compared to the simultaneous auction. However, when the order is reversed and the asymmetry between bidders is high enough the efficiency in the sequential auction is higher than in the simultaneous one.

There are only very few analytic solutions to the equilibrium bids in a simultaneous FPA with asymmetric bidders and all of them are restricted to two bidders. Thus our comparison will be restricted to cases in which we could analytically solve for the equilibrium bids in the simultaneous auction. We use results by Maskin and Riley (2000) and by Kaplan and Zamir (2012) in two different scenarios of asymmetry between two bidders. In the first scenario, stretch of probabilities, bidder 2's valuation is uniformly distributed on the interval $[0, c]$ (the weak bidder) while bidder 1's valuation is stretched to the right and is uniformly distributed on the interval $[0, c + \varepsilon]$ (the strong bidder). In the second scenario, shift of probabilities, bidder 2's valuation is again uniformly distributed on the interval $[0, c]$ (the weak bidder) while bidder 1's valuation is shifted to the right and is uniformly distributed on the interval $[\varepsilon, c + \varepsilon]$ (the strong bidder). In both cases Maskin and Riley (2000) prove that a FPA yields a higher expected revenue than a SPA and Kaplan and Zamir (2012) specify the inverse bid functions of the bidders. We show that when asymmetry is high enough (ε gets larger) a sequential bidding first price auction dominates both FPA and SPA and generates higher expected revenue.

Finally we compare the expected payoff of each of the bidders in each of the auctions and show that the ex ante expected payoff of the strong bidder is highest in the simultaneous FPA, second highest in the SPA and lowest in the sequential FPA while for the weaker bidder the order is reversed. A weak bidder prefers the sequential auction where she bids second over the simultaneous auction. Therefore a sequential first price auction may serve as a tool to increase participation of weak bidders.

Much effort has been put in the last two decades in finding equilibrium behavior in asymmetric auctions and ranking FPA and SPA in terms of revenue. Tanno (2009) proposes a necessary and sufficient condition for the existence of linear bid in the equilibrium of the asymmetric first-price auctions with two bidders and uniform distributions. Doni and Menicucci (2013) compare the expected revenue in FPA and SPA with discrete valuations with binary support. They prove that for a large set of parameters the SPA yields higher expected revenue than the FPA. Fibich, Gavious and Sela (2004) show that the difference between the expected revenue in FPA and SPA might be very small when all distributions of the bidders' valuations are on the same support and are perturbations of a common distribution. Gavious and Minchuk (2014) consider "close to uniform" distributions with identical support and show that the expected revenue in SPA may exceed that in FPA. Hubbard, Kirkegaard and Paarsch (2013) discuss numerical approximations to equilibrium behavior in asymmetric FPA. Kirkegaard (2009) suggests a new approach to examining equilibrium bids in FPA. This approach leads to identifying necessary conditions for the bid functions of the bidders to have either no crossings or a unique crossing. Kirkegaard (2011) generalizes Maskin and Riley (2000) results and identifies sufficient conditions on the distribution functions that ensure that a FPA yields higher expected revenue than a SPA. Kirkegaard (2014) proves that FPA remains optimal (over SPA) when the weak bidder's distribution falls between a shift and a stretch of the strong bidder's distribution. Mares and Swinkels (2014) define and discuss the connection between the ρ-concavity of the underlying type distributions and the bidders' bid functions. They are able to determine bounds on the equilibrium behavior in asymmetric auctions.

Not much effort has been devoted to examining auctions in which bidders bid one by one and only once. Fischer et al (2014) study a very similar model to ours. They assume an exogenous probability of a leak which is the probability that the second bidder will observe the first bidder's bid. When this probability is 1 their model coincides with ours. However they only assume symmetric bidders and therefore show (both theoretically and in an experiment) that the sequential bidding game yields less revenue than the simultaneous game. Roberts and Sweeting (2013) study a sequential bidding auction with an entry cost. In their model bidders learn their private valuations only after they decide to enter and pay the entry fee and bids can only be increased by a giving jump each round. The entry decision is therefore dependent on the entry decision of previous bidders and the bidder's own signal. They show that a sequential mechanism can give both buyers and sellers significantly higher payoffs than the commonly used simultaneous bid auction. In an earlier work, Segev and Sela (2014a) suggest a similar modification for the all-pay auction and find the perfect Baysian equilibrium of the sequential all-pay auction. However, they do not compare the expected revenue to expected revenue in the simultaneous all-pay auction. Linster (1993) studied Stackelberg rent seeking games in the form of Tullock contests where players move sequentially. He defined the perfect Bayesian equilibrium and compared the results with that of a simultaneous Tullock contest. Morgan (2003) studies sequential Tullock contests as well with incomplete information and symmetric distributions. He shows that the sequential contest may yield higher expected revenue than the simultaneous one. Finally, Ludwig (2006) compares sequential and simultaneous Tullock contests with complete and incomplete information and symmetric players. She allows for correlation between the players' types and finds that the seller prefers sequential over simultaneous contests when he aims at maximizing the expected effort sum. Moreover, the contestants may prefer sequential contests, too.

2 The Model

We consider a sequential bidding first price auction with $n \geq 2$ bidders who bid for a single indivisible object. Biders bid one by one. Bidder j, $1 \leq j \leq n$ observes the bids of bidders $1, 2, ..., j-1$ and then places a bid b_j. The winner is the bidder who placed the highest bid. We break ties in favor of later bidders. Thus, bidder j wins if her bid is larger than or equal to the bids of bidders $1, ..., j-1$ and strictly larger than the bids of bidders $j+1, ..., n$. Bidder j's valuation of the object is denoted by v_j and is private information to bidder j. It is common knowledge among the bidders that bidder j's valuation v_j is drawn from a continuos and twice differentiable distribution F_j with a positive density function f_j on the interval $[a_j, c_j]$, $0 \leq a_j < c_j$. We also assume that F_j has no atom at a_j.[3] Our aim is to characterize the perfect Bayesian equilibrium bid strategies of the bidders. In this section we do that for two bidders while in section 4 we discuss the generalization for any number of bidders.

Assume $n = 2$. We start by analyzing the second bidder's behavior. Bidder 2 observes the bid of bidder 1, b_1. Bidder 2 will then either bid b_1, if $v_2 \geq b_1$ or otherwise will bid zero. Therefore, if $c_2 \leq b_1$ then bidder 2 bids zero: $b_2(v_2; b_1) = 0$ for all $a_2 \leq v_2 \leq c_2$. If $b_1 \leq a_2$ then bidder 2 bids b_1: $b_2(v_2; b_1) = b_1$ for all $a_2 \leq v_2 \leq c_2$. Finally, if $a_2 < b_1 < c_2$ then,

$$b_2(v_2; b_1) = \begin{cases} 0 & \text{if} \quad a_2 \leq v_2 < b_1 \\ b_1 & \text{if} \quad b_1 \leq v_2 \leq c_2 \end{cases} \tag{1}$$

Bidder 1 considers the probability that bidder 2 will bid zero which is her probability of winning. If she bids less than b_2 then her payoff is zero. Otherwise, her maximization problem is given by

$$\max_{a_2 \leq b_1 \leq \min\{c_2, v_1\}} \{F_2(b_1)(v_1 - b_1)\} \tag{2}$$

We need to consider several cases. First, If $c_1 \leq a_2$ then bidder 1 can never ensure a positive payoff. In the following we assume that in this case bidder 1 bids zero. Obviously she could bid any positive bid smaller than a_2 and get the same payoff but we want to identify the PBE with the smallest possible expected revenue for the sequential auction and therefore assume she bids zero. Thus, whenever $v_1 \leq a_2$ we have $b_1(v_1) = 0$.

The first order condition for the maximization problem is given by

$$f_2(b_1)(v_1 - b_1) - F_2(b_1) = 0$$

or equivalently we can derive the inverse bid function as

$$v_1 = b_1 + \frac{F_2(b_1)}{f_2(b_1)}$$

We define the function $\Phi_2(x) = x + \frac{F_2(x)}{f_2(x)}$. We need the following condition on F_2.

[3] An atom at a_j if j is the second bidder creates discontinuity in the bidders $1, ..., j-1$ maximization problem and may lead to a case where no perfect Bayesian equilibrium exists.

Definition 1 *A continuos and twice differentiable distribution function F with a positive density f on the interval $[a, c]$ is said to be sequentially regular (SeqR) if the following function*

$$\Phi(x) = x + \frac{F(x)}{f(x)} \tag{3}$$

is strictly increasing on the interval $[a, c]$.

In the following we assume that the distribution F_2 of bidder 2's valuation is sequentially regular. This will allow us to define the bid function of bidder 1 which is the inverse function of $\Phi_2(x)$. Note that if a distribution is a convex function and sequentially regular then it is also regular in terms of the Myerson's (1981) condition (a distribution is regular if the virtual valuation $\Psi(x) = x - \frac{1-F(x)}{f(x)}$ is a strictly increasing function).[4] Moreover, if $F(x)$ is concave then it is trivially sequentially regular. Finally, note that for bidder i, $\Phi_i(a_i) = a_i$ and $\Phi_i(c_i) = c_i + \frac{1}{f_i(c_i)}$.

Proposition 1 *The following is a perfect Bayesian equilibrium (PBE) of the sequential bidding first price auction with two bidders. If $b_1 \leq a_2$ then $b_2(v_2; b_1) = b_1$ for $a_2 \leq v_2 \leq c_2$. Otherwise*

$$b_2(v_2; b_1) = \begin{cases} 0 & \text{if} \quad a_2 \leq v_2 < \min\{b_1, c_2\} \\ b_1 & \text{if} \quad \min\{b_1, c_2\} \leq v_2 \leq c_2 \end{cases}$$

If $c_1 \leq a_2$ then $b_1(v_1) = 0$. If $a_2 \leq a_1$ and $a_1 \leq \Phi_2(c_2)$ then

$$b_1(v_1) = \begin{cases} \max\left\{0, \Phi_2^{-1}(v_1)\right\} & \text{if} \quad a_1 \leq v_1 \leq \min\{\Phi_2(c_2), c_1\} \\ c_2 & \text{if} \quad \min\{\Phi_2(c_2), c_1\} \leq v_1 \leq c_1 \end{cases}$$

If $\Phi_2(c_2) \leq a_1$ then

$$b_1(v_1) = c_2 \quad \text{for all } a_1 \leq v_1 \leq c_1$$

Finally, if $a_1 < a_2$ then

$$b_1(v_1) = \begin{cases} 0 & \text{if} \quad a_1 \leq v_1 < a_2 \\ \max\left\{0, \Phi_2^{-1}(v_1)\right\} & \text{if} \quad a_2 \leq v_1 \leq \min\{\Phi_2(c_2), c_1\} \\ c_2 & \text{if} \quad \min\{\Phi_2(c_2), c_1\} \leq v_1 \leq c_1 \end{cases}$$

Proof. *Bidder 2 best responds to bidder 1's behavior since he places the same bid as bidder 1 whenever it is below his valuation. Bidder 1 solves the maximization problem (2). This together with the assumption that if $v_1 \leq a_2$ then bidder 1 bids zero since she cannot ensure any positive payoff given bidder 2's strategy (as explained above), gives us the described PBE.* ■

Using proposition 1 we can express the expected revenue of the seller in this PBE. Note that with only two bidders the expected revenue of the seller is equal to the expected bid of the first bidder (the second

[4] Since if $\Phi(x) = \Psi(x) + \frac{1}{f(x)}$ is strictly increasing then $\frac{d}{dx}\Phi(x) = \frac{d}{dx}\Psi(x) - \frac{f'(x)}{(f(x))^2} > 0$ and the convexity of $F(x)$ implies $\frac{d}{dx}\Psi(x) > \frac{f'(x)}{(f(x))^2} > 0$.

bidder either places the same bid or bids zero. Therefore the expected revenue of the seller is given by

$$R_{\text{seq}}^{FPA}=\begin{cases} 0 & \text{if} \quad c_1 \le a_2 \\ \int_{a_1}^{c_1}\left(\max\left\{0,\Phi_2^{-1}\left(v_1\right)\right\}\right) f_1\left(v_1\right) dv_1 & \text{if} \quad a_2 \le a_1 < c_1 \le \Phi_2\left(c_2\right) \\ \int_{a_1}^{\Phi_2(c_2)}\left(\max\left\{0,\Phi_2^{-1}\left(v_1\right)\right\}\right) f_1\left(v_1\right) dv_1 + c_2\left(1-F_1\left(\Phi_2\left(c_2\right)\right)\right) & \text{if} \quad a_2 \le a_1 < \Phi_2\left(c_2\right) \le c_1 \\ c_2 & \text{if} \quad a_2 < \Phi_2\left(c_2\right) \le a_1 < c_1 \\ \int_{a_2}^{c_1}\left(\max\left\{0,\Phi_2^{-1}\left(v_1\right)\right\}\right) f_1\left(v_1\right) dv_1 & \text{if} \quad a_1 < a_2 \le c_1 \le \Phi_2\left(c_2\right) \\ \int_{a_2}^{\Phi_2(c_2)}\left(\max\left\{0,\Phi_2^{-1}\left(v_1\right)\right\}\right) f_1\left(v_1\right) dv_1 + c_2\left(1-F_1\left(\Phi_2\left(c_2\right)\right)\right) & \text{if} \quad a_1 \le a_2 < \Phi_2\left(c_2\right) \le c_1 \end{cases}$$

The following example is with uniform distributions and it applies to the first two scenarios we examine below.

Example 1 *Assume that* v_1 *is uniformly distributed on* $[a_1, c_1]$ *and* v_2 *is uniformly distributed on* $[a_2, c_2]$ *then* F_2 *is sequentially regular and* $\Phi_2(x) = 2x - a_2$ *and* $\Phi_2^{-1}(x) = \frac{1}{2}(x + a_2)$*. Therefore, for any* v_1 *we have* $\max\left\{0, \Phi_2^{-1}(v_1)\right\} = \frac{1}{2}(v_1 + a_2)$ *and then*

$$R_{\text{seq}}^{FPA}=\begin{cases} 0 & \textit{if} \quad c_1 \le a_2 \\ \frac{1}{4}\left(a_1+2a_2+c_1\right) & \textit{if} \quad a_2 \le a_1 < c_1 \le 2c_2 - a_2 \\ \frac{1}{4(c_1-a_1)}\left(4c_2\left(a_2+c_1-c_2\right)-\left(a_1+a_2\right)^2\right) & \textit{if} \quad a_2 \le a_1 < 2c_2 - a_2 \le c_1 \\ c_2 & \textit{if} \quad a_2 < 2c_2 - a_2 \le a_1 < c_1 \\ \frac{1}{4(c_1-a_1)}\left(c_1-a_2\right)\left(3a_2+c_1\right) & \textit{if} \quad a_1 < a_2 \le c_1 \le 2c_2 - a_2 \\ \frac{1}{c_1-a_1}\left(\left(a_2+c_1\right)c_2 - a_2^2 - c_2^2\right) & \textit{if} \quad a_1 \le a_2 < 2c_2 - a_2 \le c_1 \end{cases}$$

In the next section we use the above results to compare the expected revenue in the PBE of the sequential bidding first price auction with the expected revenue of a simultaneous bidding first price and second price auctions. We follow Maskin and Riley (2000) and examine two cases of asymmetry while using Kaplan and Zamir (2012) to derive the explicit bidding functions of the bidders. In both cases we have a strong bidder and a weak bidder. We say that a bidder is stronger than another bidder if the distribution from which her valuation is drawn, first order stochastically dominates that of the other. When examining the sequential bidding auction we would therefore first need to determine what is the optimal (in terms of the seller's revenue) order of the bidders.

3 Ranking the auctions

For simultaneous bidding first price auction there are only limited results for the Bayesian equilibrium behavior of bidders. Kaplan and Zamir (2012) find the equilibrium behavior of two bidders with uniform distributions and general supports. Therefore we restrict attention to uniform distributions in order to be able to derive an explicit expression for the expected revenue in the Bayesian equilibrium of the simultaneous FPA.

3.1 Stretch of probabilities

Assume that F_1 is a uniform distribution on $[0, c+\varepsilon]$, for $\varepsilon > 0$ while F_2 is uniform on $[0, c]$. Bidder 1 is therefore the strong bidder. If bidder 1 bids first then we have

$$R_{\text{seq}}^{FPA}(1,2) = \begin{cases} \frac{1}{4}(c+\varepsilon) & \text{if} \quad 0 \le \varepsilon \le c \\ \frac{c\varepsilon}{(c+\varepsilon)} & \text{if} \quad c \le \varepsilon \end{cases}$$

while if bidder 1 bids second then

$$R_{\text{seq}}^{FPA}(2,1) = \frac{1}{4}c$$

Therefore the optimal order is when the stronger bidder bids first. We thus assume in the following that we are able to determine the order of the bidders and we let the weaker bidder bid second. Then we have

$$R_{\text{seq}}^{FPA} = \begin{cases} \frac{1}{4}(c+\varepsilon) & \text{if} \quad 0 \le \varepsilon \le c \\ \frac{c\varepsilon}{(c+\varepsilon)} & \text{if} \quad c \le \varepsilon \end{cases}$$

We use Kaplan and Zamir (2012) to derive the equilibrium bid function of the simultaneous bid FPA. The inverse bid functions, $v_1(b)$ and $v_2(b)$ solve the following set of differential equations:

$$\begin{aligned} \frac{f_2(v_2(b))}{F_2(v_2(b))} v_2'(b) &= \frac{1}{v_1(b) - b} \\ \frac{f_1(v_1(b))}{F_1(v_1(b))} v_1'(b) &= \frac{1}{v_2(b) - b} \end{aligned} \tag{4}$$

with the following boundary conditions: $v_1\left(\frac{c(c+\varepsilon)}{2c+\varepsilon}\right) = c+\varepsilon$ and $v_2\left(\frac{c(c+\varepsilon)}{2c+\varepsilon}\right) = c$ and $v_1(0) = v_2(0) = 0$. Therefore we have

$$\begin{aligned} v_2(b) &= \frac{2bc^2(\varepsilon+c)^2}{c^2(\varepsilon+c)^2 + \varepsilon(\varepsilon+2c)b^2} \\ v_1(b) &= \frac{2bc^2(\varepsilon+c)^2}{\left(c^2(\varepsilon+c)^2 - \varepsilon(\varepsilon+2c)b^2\right)} \end{aligned}$$

Or

$$\begin{aligned} b_1(v) &= \frac{c(\varepsilon+c)\left(-c(\varepsilon+c) + \sqrt{c^2(\varepsilon+c)^2 + \varepsilon(\varepsilon+2c)v^2}\right)}{\varepsilon(\varepsilon+2c)v} \\ b_2(v) &= \frac{c(\varepsilon+c)\left(c(\varepsilon+c) - \sqrt{c^2(\varepsilon+c)^2 - \varepsilon v^2(\varepsilon+2c)}\right)}{\varepsilon v(\varepsilon+2c)} \end{aligned}$$

Note that for every $v \in [0, c]$ we have $b_2(v) > b_1(v)$. The weaker bidder bids more aggressively in the equilibrium of the simultaneous FPA. For example, for $c = 1$ and $\varepsilon = \frac{1}{4}$ the following figure describes the bid functions of the bidders.

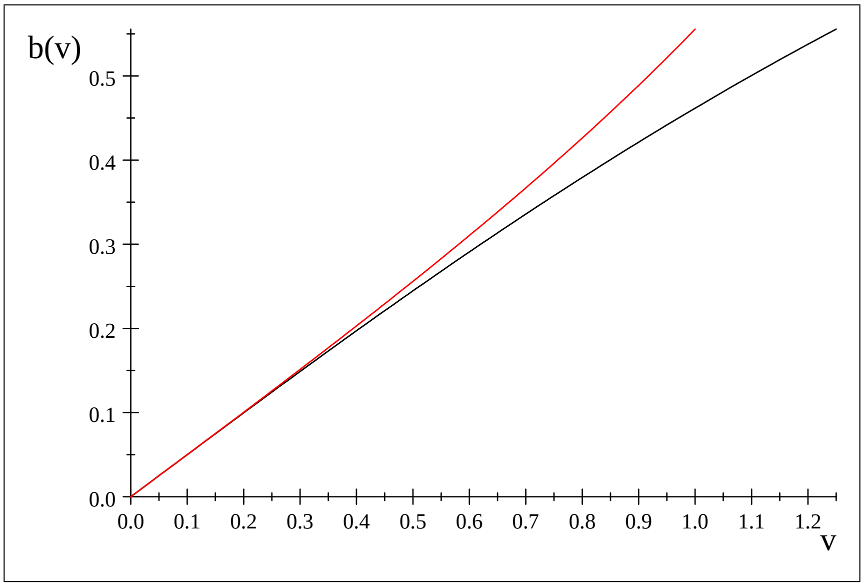

figure 1: Eq. bid functions, $b_1(v)$ in black and $b_2(v)$ in red.

In order to find the expected revenue of the seller in the simultaneous FPA we define the random variable of the winning bid P. This random variable is distributed on $\left[0, \frac{c(c+\varepsilon)}{2c+\varepsilon}\right]$ according to the following distribution function

$$G(p) = \Pr(\max\{b_1(v), b_2(v)\} \leq p) = F_1(v_1(p)) F_2(v_2(p))$$

with a density function

$$g(p) = \frac{8pc^3(\varepsilon+c)^3\left(\varepsilon^2(\varepsilon+2c)^2 p^4 + c^4(\varepsilon+c)^4\right)}{\left(c^4(\varepsilon+c)^4 - \varepsilon^2(\varepsilon+2c)^2 p^4\right)^2}$$

Therefore we have

$$\begin{aligned} R_{sim}^{FPA} &= \int_0^{\frac{c(c+\varepsilon)}{2c+\varepsilon}} pg(p)\,dp \\ &= \frac{c^2(\varepsilon+c)^2\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)} + 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)\right)}{\varepsilon(\varepsilon+2c)\sqrt{\varepsilon(\varepsilon+2c)}} + \frac{c(\varepsilon+c)}{(\varepsilon+2c)} \end{aligned}$$

Finally, in the SPA the bidders bid their valuation and then the random variable P is distributed on $[0, c]$ with the following distribution function

$$K(p) = \Pr(\min\{v_1, v_2\} \leq p) = p\frac{(\varepsilon+2c-p)}{c(\varepsilon+c)}$$

and a density function

$$k(p) = \frac{\varepsilon+2c-2p}{c(\varepsilon+c)}$$

and therefore

$$R^{SPA} = \int_0^c pk(p)\,dp = \frac{1}{6}c\frac{3\varepsilon+2c}{\varepsilon+c}$$

We are now ready to compare the auctions in terms of expected revenue. We already know from Maskin and Riley (2000) that the expected revenue from the second price auction is always lower than the expected revenue from the simultaneous first price auction. In other words $R^{SPA}(\varepsilon) < R^{FPA}_{sim}(\varepsilon)$. We have the following two propositions

Proposition 2 *For every c, there exists a cut-off* $0 < \varepsilon^*(c) \leq c$ *such that for all* $\varepsilon < \varepsilon^*(c), R^{FPA}_{sim} > R^{FPA}_{\text{seq}}$ *and for all* $\varepsilon > \varepsilon^*(c), R^{FPA}_{sim} < R^{FPA}_{\text{seq}}$. *Moreover, this cutoff is increasing in c.*

Proof. *In the appendix* ■

Proposition 3 *For every c, there exists a cut-off* $0 < \varepsilon^{**}(c) < c$ *such that for all* $\varepsilon < \varepsilon^{**}(c), R^{SPA} > R^{FPA}_{\text{seq}}$ *and for all* $\varepsilon > \varepsilon^{**}(c), R^{SPA} < R^{FPA}_{\text{seq}}$. *The cutoff* $\varepsilon^{**}(c)$ *is increasing in c and* $\varepsilon^{**}(c) < \varepsilon^*(c)$.

Proof. *In the appendix* ■

For example, for $c = 1$ we plot the expected revenue in all three auctions as a function of ε in the following figure

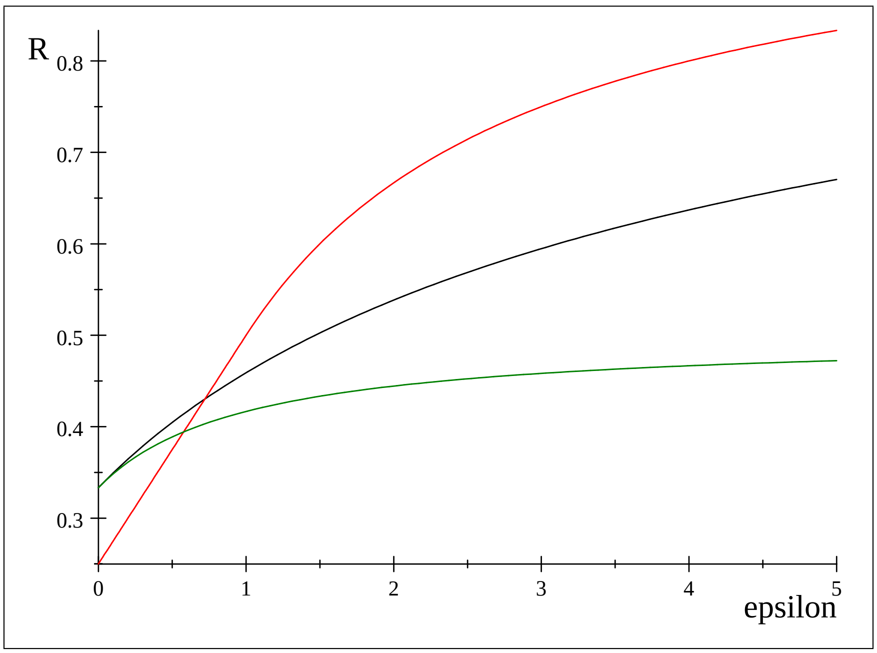

Figure 2: R^{SPA} in green, R^{FPA}_{sim} in black, R^{FPA}_{seq} in red

We therefore conclude that when the asymmetry is high enough, $\varepsilon > \varepsilon^*(c)$ then the sequential bidding auction yields higher expected revenue than both simultaneous auctions. Note that $\varepsilon^*(c) < c$ and therefore the asymmetry need not be so large in order to conclude that the sequential bidding auction dominates the simultaneous auctions.

In terms of efficiency, however, the sequential bidding auction always yields lower efficiency than the simultaneous auctions if the stronger bidder bids first. We define efficiency as the probability that the winner is the bidder with the higher valuation. Recall that the simultaneous first price auction does not guarantee efficiency when bidders are asymmetric since the weaker bidder bids more aggressively than the stronger one (less shading down bids) and then the weaker bidder may win although her valuation is lower than the stronger bidder's valuation. This phenomenon is also present in the sequential first price auction. When the weaker bidder bids second she has an advantage over the stronger one since she gets to observe his bid and can win by placing the same bid. This leads to cases where the weaker bidder wins although her valuation is lower than the stronger bidder's valuation. In the following we compare the efficiency of the mechanisms. For the second price auction we have

$$Eff^{SPA} = 1$$

For the simultaneous first price auction

$$\begin{aligned} Eff_{sim}^{FPA} &= 1 - \Pr\left(b_2\left(v_2\right) > b_1\left(v_1\right) \text{ and } v_1 > v_2\right) \\ &= 1 - \int_0^c \left(\int_{v_2}^{\frac{cv_2(\varepsilon+c)}{\sqrt{\left(c^2(\varepsilon+c)^2 - \varepsilon(\varepsilon+2c)v_2^2\right)}}} \frac{1}{c_1+\varepsilon} dv_1 \right) \frac{1}{c} dv_2 = \frac{1}{2} \frac{2\varepsilon^2 + 4c^2 + 5\varepsilon c}{(\varepsilon+c)(\varepsilon+2c)} \end{aligned}$$

For the sequential FPA, when the stronger bidder bids first, we have, for $\varepsilon \leq c$

$$\begin{aligned} Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c]) &= 1 - \Pr\left(b_1\left(v_1\right) < v_2 \text{ and } v_2 < v_1\right) \\ & 1 - \left(\int_0^c \left(\int_{\frac{1}{2}v_1}^{v_1} \frac{1}{c} dv_2 \right) \frac{1}{c+\varepsilon} dv_1 + \int_c^{c+\varepsilon} \left(\int_{\frac{1}{2}v_1}^{c} \frac{1}{c} dv_2 \right) \frac{1}{(c+\varepsilon)} dv_1 \right) \\ &= \frac{1}{4} \frac{\varepsilon^2 + 3c^2 + 2\varepsilon c}{c(\varepsilon+c)} \end{aligned}$$

while for $\varepsilon \geq c$

$$\begin{aligned} Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c]) &= 1 - \Pr\left(b_1\left(v_1\right) < v_2 \text{ and } v_2 < v_1\right) \\ & 1 - \left(\int_0^c \left(\int_{\frac{1}{2}v_1}^{v_1} \frac{1}{c} dv_2 \right) \frac{1}{c+\varepsilon} dv_1 + \int_c^{2c} \left(\int_{\frac{1}{2}v_1}^{c} \frac{1}{c} dv_2 \right) \frac{1}{c+\varepsilon} dv_1 \right) \\ &= \frac{1}{2} \frac{2\varepsilon + c}{\varepsilon + c} \end{aligned}$$

Therefore we have for every c and every ε

$$Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c]) < Eff_{sim}^{FPA} < Eff^{SPA} = 1$$

However, if we order the bidders such that the weaker bidder bids first than we can increase efficiency. In this case we have

$$b_2\left(v_2; b_1\right) = \begin{cases} 0 & \text{if} \quad 0 \leq v_2 < b_1 \\ b_1 & \text{if} \quad b_1 \leq v_2 < c+\varepsilon \end{cases}$$

and

$$b_1\left(v_1\right) = \frac{1}{2} v_1 \text{ for all } 0 \leq v_1 \leq c$$

and

$$\begin{aligned} Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon]) &= 1 - \Pr\left(b_1\left(v_1\right) < v_2 \text{ and } v_2 < v_1\right) \\ &= 1 - \left(\int_0^c \left(\int_{\frac{1}{2}v_1}^{v_1} \frac{1}{c+\varepsilon} dv_2 \right) \frac{1}{c} dv_1 \right) = \frac{1}{4} \frac{3c + 4\varepsilon}{c+\varepsilon} \end{aligned}$$

Then

$$Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c]) < Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon])$$

and for $\varepsilon < 2c$ we have

$$Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon]) < Eff_{sim}^{FPA}$$

while for $\varepsilon > 2c$ we have

$$Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon]) > Eff_{sim}^{FPA}$$

For example, for $c = 1$ the following figure describes the efficiency as a function of ε. In the figure Eff^{SPA} appears in yellow, Eff_{sim}^{FPA} in black, $Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c])$ in red and $Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon])$ in light green

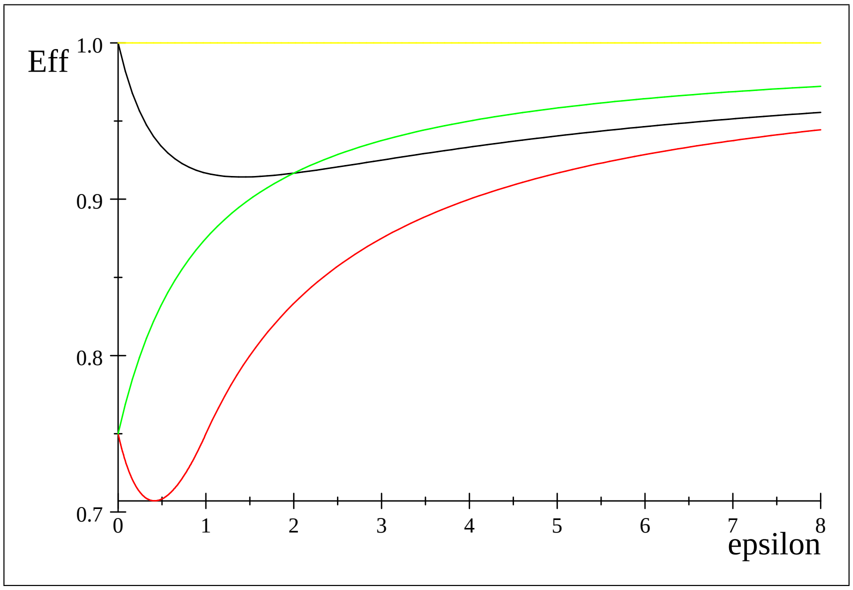

Figure 3

When the asymmetry is very large the efficiency in all auctions approaches 1. We have

$$\lim_{\varepsilon\to\infty}\left(Eff_{\text{seq}}^{FPA}([0,c],[0,c+\varepsilon])\right) = \lim_{\varepsilon\to\infty}\left(Eff_{\text{seq}}^{FPA}([0,c+\varepsilon],[0,c])\right) = \lim_{\varepsilon\to\infty}\left(Eff_{\text{seq}}^{FPA}\right) = 1$$

Finally we discuss the differences between the auctions in the eyes of the bidders. For each of the auctions we calculate the expected revenue of the strong bidder and the weak bidder. For the simultaneous SPA we have

$$\begin{aligned} u_1^{SPA}(v) &= \begin{cases} \int_0^v (v-v_2)\frac{1}{c}dv_2 = \frac{1}{2}\frac{v^2}{c} & \text{if} \quad 0 \le v \le c \\ \int_0^c (v-v_2)\frac{1}{c}dv_2 = v - \frac{1}{2}c & \text{if} \quad c \le v \le c+\varepsilon \end{cases} \\ u_2^{SPA}(v) &= \int_0^v (v-v_1)\frac{1}{c+\varepsilon}dv_1 = \frac{1}{2}\frac{v^2}{c+\varepsilon} \end{aligned}$$

and ex ante we have

$$
\begin{aligned}
U_1^{SPA} &= \int_0^c \left(\frac{1}{2}\frac{v^2}{c}\right)\frac{1}{c+\varepsilon}dv + \int_c^{c+\varepsilon}\left(v-\frac{1}{2}c\right)\frac{1}{c+\varepsilon}dv = \frac{1}{6}\frac{3c\varepsilon+3\varepsilon^2+c^2}{c+\varepsilon} \\
U_2^{SPA} &= \int_0^c \left(\frac{1}{2}\frac{v^2}{c+\varepsilon}\right)\frac{1}{c}dv = \frac{1}{6}\frac{c^2}{c+\varepsilon}
\end{aligned}
$$

For the simultaneous FPA we have

$$
\begin{aligned}
u_1^{simFPA}(v) &= \int_0^{\frac{cv(c+\varepsilon)}{\sqrt{(c^2(c+\varepsilon)^2+v^2\varepsilon(2c+\varepsilon))}}} (v-b_1(v))\frac{1}{c}dv_2 \\
&= \frac{(c+\varepsilon)\left(\varepsilon(\varepsilon+2c)v^2+c^2(\varepsilon+c)^2-c(\varepsilon+c)\sqrt{c^2(\varepsilon+c)^2+\varepsilon(\varepsilon+2c)v^2}\right)}{\varepsilon(\varepsilon+2c)\sqrt{\left(c^2(c+\varepsilon)^2+v^2\varepsilon(2c+\varepsilon)\right)}} \\
u_2^{simFPA}(v) &= \int_0^{\frac{cv(c+\varepsilon)}{\sqrt{(c^2(c+\varepsilon)^2-v^2\varepsilon(2c+\varepsilon))}}} (v-b_2(v))\frac{1}{c+\varepsilon}dv_1 \\
&= \frac{c\left(\varepsilon v^2(\varepsilon+2c)-c^2(\varepsilon+c)^2+c(\varepsilon+c)\sqrt{c^2(\varepsilon+c)^2-\varepsilon v^2(\varepsilon+2c)}\right)}{\varepsilon(\varepsilon+2c)\sqrt{\left(c^2(c+\varepsilon)^2-v^2\varepsilon(2c+\varepsilon)\right)}}
\end{aligned}
$$

and ex ante we have

$$
\begin{aligned}
U_1^{simFPA} &= \int_0^{c+\varepsilon}\left(\frac{(c+\varepsilon)\left(\varepsilon(\varepsilon+2c)v^2+c^2(\varepsilon+c)^2-c(\varepsilon+c)\sqrt{c^2(\varepsilon+c)^2+\varepsilon(\varepsilon+2c)v^2}\right)}{\varepsilon(\varepsilon+2c)\sqrt{\left(c^2(c+\varepsilon)^2+v^2\varepsilon(2c+\varepsilon)\right)}}\right)\frac{1}{c+\varepsilon}dv \\
&= \frac{(c+\varepsilon)^2}{2\varepsilon(2c+\varepsilon)}\left(\frac{c^2}{\sqrt{\varepsilon(2c+\varepsilon)}}\left(3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\,\mathrm{arccosh}\left(\frac{c+\varepsilon}{c}\right)\right)-(c-\varepsilon)\right) \\
U_2^{simFPA} &= \int_0^c\left(\frac{c\left(\varepsilon v^2(\varepsilon+2c)-c^2(\varepsilon+c)^2+c(\varepsilon+c)\sqrt{c^2(\varepsilon+c)^2-\varepsilon v^2(\varepsilon+2c)}\right)}{\varepsilon(\varepsilon+2c)\sqrt{\left(c^2(c+\varepsilon)^2-v^2\varepsilon(2c+\varepsilon)\right)}}\right)\frac{1}{c}dv \\
&= \frac{c^2(c+\varepsilon)^2}{2\varepsilon(2c+\varepsilon)\sqrt{\varepsilon(2c+\varepsilon)}}\left(\frac{1}{2}\pi-\arcsin\left(\frac{c}{c+\varepsilon}\right)-2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)\right)+\frac{1}{2}\frac{c^2(c+2\varepsilon)}{\varepsilon(2c+\varepsilon)}
\end{aligned}
$$

Finally, for the sequential FPA we have, when $0 \leq \varepsilon \leq c$

$$
\begin{aligned}
u_1^{\mathrm{seq}\,FPA}(v) &= \frac{v^2}{4c} \\
u_2^{\mathrm{seq}\,FPA}(v) &= \begin{cases} \int_0^{2v}\left(v-\frac{1}{2}v_1\right)\frac{1}{c+\varepsilon}dv_1 = \frac{v^2}{c+\varepsilon} & \text{if} \quad 0 \leq v \leq \frac{1}{2}(c+\varepsilon) \\ \int_0^{c+\varepsilon}\left(v-\frac{1}{2}v_1\right)\frac{1}{c+\varepsilon}dv_1 = v-\frac{1}{4}(c+\varepsilon) & \text{if} \quad \frac{1}{2}(c+\varepsilon) \leq v \leq c \end{cases}
\end{aligned}
$$

and when $c \leq \varepsilon$ then

$$
\begin{aligned}
u_1^{\mathrm{seq}\,FPA}(v) &= \begin{cases} \frac{v^2}{4c} & \text{if} \quad 0 \leq v \leq 2c \\ v-c & \text{if} \quad 2c \leq v \leq c+\varepsilon \end{cases} \\
u_2^{\mathrm{seq}\,FPA}(v) &= \frac{v^2}{c+\varepsilon}
\end{aligned}
$$

Note that although the second bidder is the weaker one she enjoys a higher expected revenue for every type since she has the benefit of bidding second. The ex ante expected payoff is given by

$$U_1^{\text{seq}\,FPA} = \begin{cases} \frac{1}{12}\frac{(c+\varepsilon)^2}{c} & \text{if} \quad 0 \leq \varepsilon \leq c \\ \frac{1}{6}\frac{3\varepsilon^2+c^2}{c+\varepsilon} & \text{if} \quad c \leq \varepsilon \end{cases}$$

$$U_2^{\text{seq}\,FPA} = \begin{cases} \frac{1}{24}\frac{7c^2-4c\varepsilon+\varepsilon^2}{c} & \text{if} \quad 0 \leq \varepsilon \leq c \\ \frac{1}{3}\frac{c^2}{c+\varepsilon} & \text{if} \quad c \leq \varepsilon \end{cases}$$

We prove the following proposition

Proposition 4 *For every c and ε we have*

$$U_1^{\text{seq}\,FPA} < U_1^{simFPA} < U_1^{SPA}$$

and

$$U_2^{SPA} < U_2^{simFPA} < U_2^{\text{seq}\,FPA}$$

Proof. *In the appendix* ■

The above proposition establishes the fact that the weaker bidder prefers the sequential bidding auction over the two simultaneous bidding auctions while the stronger one does not. Therefore letting the weaker bidder bid after the stronger bidder bids, both increases her ex-ante expected payoff and the expected revenue of the seller (when the asymmetry is large enough).

For example, when $c = 1$ the next two figures present the ex ante expected payoff of the strong bidder, bidder 1 (figure 4) and the weak bidder, bidder 2 (figure 5) in all three auctions

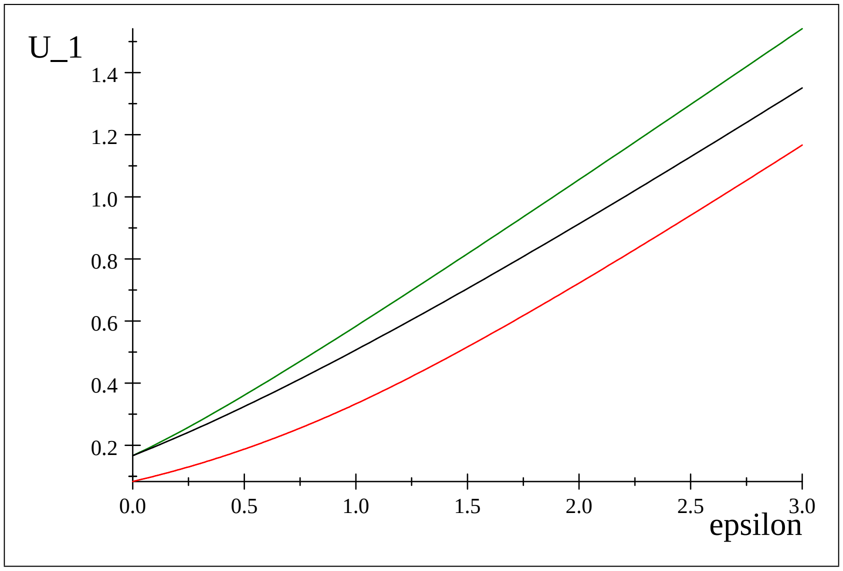

Figure 4: U_1^{SPA} in green, U_1^{simFPA} in black, $U_1^{\text{seq}\,FPA}$ in red

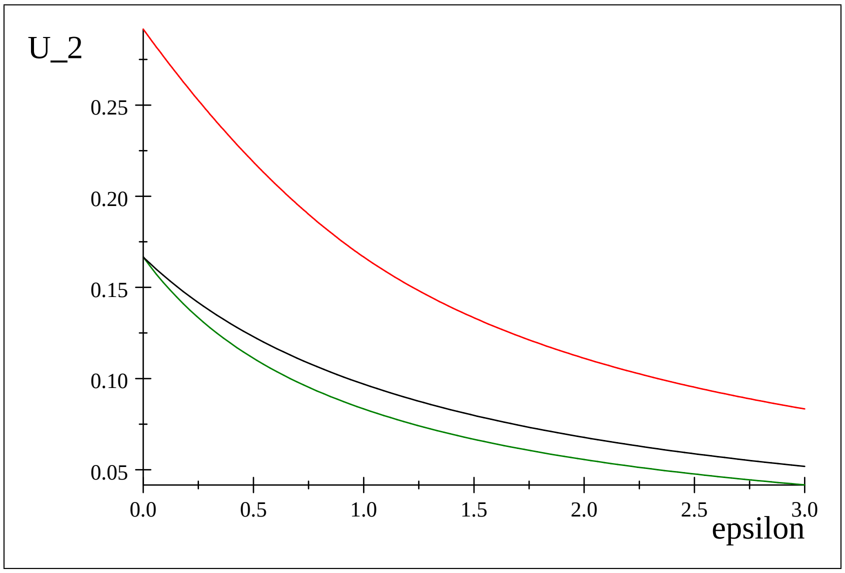

Figure 5: U_2^{SPA} in green, U_2^{simFPA} in black, $U_2^{\text{seq}\,FPA}$ in red

3.2 Shift of probabilities

Assume that F_1 is a uniform distribution on $[\varepsilon, c+\varepsilon]$ while F_2 is uniform on $[0, c]$. Bidder 1 is therefore the strong bidder. If bidder 1 bids first then we have

$$R_{\text{seq}}^{FPA}(1,2) = \begin{cases} \int_{\varepsilon}^{c+\varepsilon} \left(\frac{1}{2}v\right) \frac{1}{c} dv = \frac{1}{4}(c+2\varepsilon) & \text{if} \quad 0 \le \varepsilon \le c \\ \int_{\varepsilon}^{2c} \left(\frac{1}{2}v\right) \frac{1}{c} dv + \int_{2c}^{c+\varepsilon} c \frac{1}{c} dv = \frac{1}{4}\varepsilon \frac{4c-\varepsilon}{c} & \text{if} \quad c \le \varepsilon \le 2c \\ c & \text{if} \quad 2c \le \varepsilon \end{cases}$$

while if bidder 1 bids second then

$$R_{\text{seq}}^{FPA}(2,1) = \begin{cases} \int_{\varepsilon}^{c} \frac{1}{2}(v+\varepsilon) \frac{1}{c} dv = \frac{1}{4c}(c-\varepsilon)(c+3\varepsilon) & \text{if} \quad 0 \le \varepsilon \le c \\ 0 & \text{if} \quad c \le \varepsilon \end{cases}$$

Therefore the optimal order is again for the stronger bidder to bid first. We thus assume in the following that we are able to determine the order of the bidders and we let the weaker bidder bid second. Then we have

$$R_{\text{seq}}^{FPA} = \begin{cases} \int_{\varepsilon}^{c+\varepsilon} \left(\frac{1}{2}v\right) \frac{1}{c} dv = \frac{1}{4}(c+2\varepsilon) & \text{if} \quad 0 \le \varepsilon \le c \\ \int_{\varepsilon}^{2c} \left(\frac{1}{2}v\right) \frac{1}{c} dv + \int_{2c}^{c+\varepsilon} c \frac{1}{c} dv = \frac{1}{4}\varepsilon \frac{4c-\varepsilon}{c} & \text{if} \quad c \le \varepsilon \le 2c \\ c & \text{if} \quad 2c \le \varepsilon \end{cases}$$

We again use Kaplan and Zamir (2012) to derive the equilibrium bid function of the simultaneous bid FPA. We first note that if $\varepsilon \ge 2c$ the stronger bidder will ensure winning by bidding c therefore the following is an equilibrium of the simultaneous FPA:

$$\begin{aligned} b_1(v_1) &= c \\ b_2(v_2) &= v_2 \end{aligned}$$

and then

$$R_{sim}^{FPA} = \int_{\varepsilon}^{c+\varepsilon} c\frac{1}{c}dv_1 = c$$

When $0 \leq \varepsilon \leq 2c$ the inverse bid functions, $v_1(b)$ and $v_2(b)$ solve the same set of differential equations (4) as before, with the following boundary conditions: $v_1\left(\frac{1}{8}\frac{4c\varepsilon-\varepsilon^2+4c^2}{c}\right) = c+\varepsilon$ and $v_2\left(\frac{1}{8}\frac{4c\varepsilon-\varepsilon^2+4c^2}{c}\right) = c$ and $v_1\left(\frac{\varepsilon}{2}\right) = \varepsilon$ and $v_2\left(\frac{\varepsilon}{2}\right) = \frac{\varepsilon}{2}$.[5] Therefore we have

$$\begin{aligned} v_2(b) &= \varepsilon + \frac{\varepsilon^2(2c-\varepsilon)}{2\left((2b-\varepsilon)(2c+\varepsilon)e^{\frac{\varepsilon}{2b-\varepsilon}-\frac{4\varepsilon c}{(2c-\varepsilon)(2c+\varepsilon)}} - 2b(2c-\varepsilon)\right)} \\ v_1(b) &= \frac{\varepsilon^2(2c+\varepsilon)}{2\left((2b-\varepsilon)(2c-\varepsilon)e^{\frac{4\varepsilon c}{(2c-\varepsilon)(2c+\varepsilon)}-\frac{\varepsilon}{2b-\varepsilon}} + 2(\varepsilon-b)(2c+\varepsilon)\right)} \end{aligned}$$

Again, for every $b \in \left[\frac{\varepsilon}{2}, \frac{1}{8}\frac{4c\varepsilon-\varepsilon^2+4c^2}{c}\right]$ we have $v_2(b) < v_1(b)$. The weaker bidder bids more aggressively in the equilibrium of the simultaneous FPA. We again define the random variable of the winning bid P. This random variable is distributed on $\left[\frac{\varepsilon}{2}, \frac{1}{8}\frac{4c\varepsilon-\varepsilon^2+4c^2}{c}\right]$ according to the following distribution function

$$G(p) = \Pr(\max\{b_1(v), b_2(v)\} \leq p) = F_1(v_1(p))F_2(v_2(p))$$

with a density function

$$g(p) = \frac{(2c-\varepsilon)(2c+\varepsilon)\varepsilon^4 e^{\frac{4\varepsilon c}{(2c-\varepsilon)(2c+\varepsilon)}}}{2c^2}\frac{e^{\frac{\varepsilon}{2p-\varepsilon}}}{(2p-\varepsilon)}\left(\begin{array}{c} \frac{1}{\left(-2p(2c-\varepsilon)e^{\frac{4\varepsilon c}{(2c-\varepsilon)(2c+\varepsilon)}}+(2p-\varepsilon)(2c+\varepsilon)e^{\frac{\varepsilon}{2p-\varepsilon}}\right)^2} \\ +\frac{1}{\left((-2p+\varepsilon)(2c-\varepsilon)e^{\frac{4\varepsilon c}{(2c-\varepsilon)(2c+\varepsilon)}}+2(p-\varepsilon)(2c+\varepsilon)e^{\frac{\varepsilon}{2p-\varepsilon}}\right)^2} \end{array}\right)$$

Therefore we have

$$R_{sim}^{FPA} = \int_{\frac{\varepsilon}{2}}^{\frac{1}{8}\frac{4c\varepsilon-\varepsilon^2+4c^2}{c}} pg(p)\,dp$$

Unfortunately we could not derive here the analytic expression of the revenue. However, we could numerically evaluate the integral for any set of parameters (c and ε).

Finally, in the SPA the bidders bid their valuation and then the random variable P is distributed on $[0, c]$ with the following distribution function for $0 \leq \varepsilon \leq c$

$$K(p) = \Pr(\min\{v_1, v_2\} \leq p) = \begin{cases} \frac{p}{c} & \text{if } 0 \leq p \leq \varepsilon \\ \frac{2cp-c\varepsilon+p\varepsilon-p^2}{c^2} & \text{if } \varepsilon \leq p \leq c \end{cases}$$

Then

$$k(p) = \begin{cases} \frac{1}{c} & \text{if } 0 \leq p \leq \varepsilon \\ \frac{2c-2p+\varepsilon}{c^2} & \text{if } \varepsilon \leq p \leq c \end{cases}$$

and

$$R^{SPA} = \int_0^{\varepsilon}\frac{p}{c}dp + \int_{\varepsilon}^{c}\left(\frac{2c-2p+\varepsilon}{c^2}\right)pdp = \frac{1}{6}\frac{\varepsilon^3-3c\varepsilon^2+3c^2\varepsilon+2c^3}{c^2}$$

[5] The equilibrium is such that the weaker bidder bids her true valuation up to $\frac{\varepsilon}{2}$ and from there on bids according to $b_2(v) = v_2^{-1}(b)$.

For $c \leq \varepsilon$, we have for $0 \leq p \leq c$

$$K(p) = \Pr(\min\{v_1, v_2\} \leq p) = \frac{p}{c}$$

Then

$$k(p) = \frac{1}{c}$$

and

$$R^{SPA} = \int_0^c \frac{p}{c} dp = \frac{1}{2}c$$

We are now ready to compare the auctions in terms of expected revenue. We already know from Maskin and Riley (2000) that the expected revenue from the simultaneous second price auction is always lower than the expected revenue from the simultaneous first price auction. In other words $R^{SPA}(\varepsilon) < R^{FPA}_{sim}(\varepsilon)$. We have the following proposition.

Proposition 5 *For every c, there exists a cut-off $0 < \varepsilon^{**}(c) < c$ such that for all $\varepsilon < \varepsilon^{**}(c)$, $R^{SPA} > R^{FPA}_{\text{seq}}$ and for all $\varepsilon > \varepsilon^{**}(c)$, $R^{SPA} < R^{FPA}_{\text{seq}}$. The cutoff $\varepsilon^{**}(c)$ is increasing in c.*

Proof. *In the appendix* ■

For example, for $c = 1$, figure 6 presents R^{FPA}_{seq} and R^{SPA}

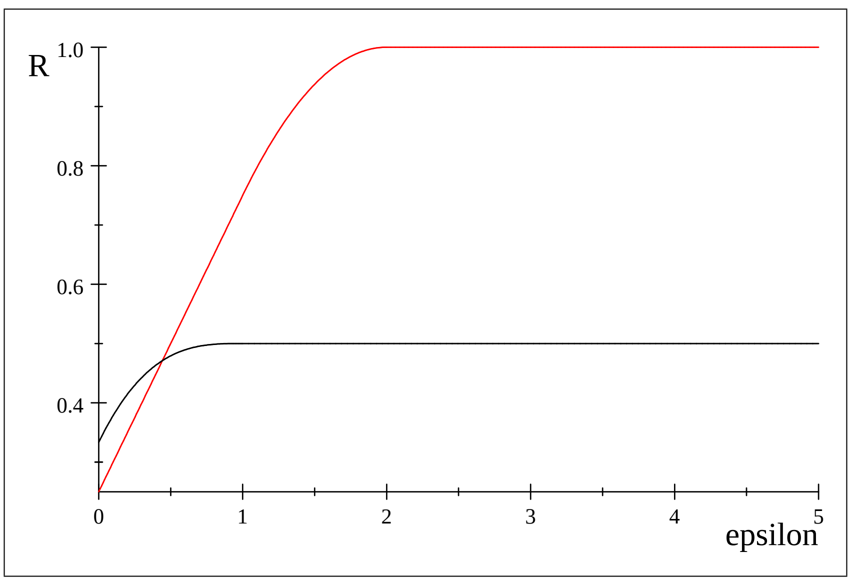

Figure 6: R^{FPA}_{seq} in red and R^{SPA} in black

Since we do not have the explicit expression for the revenue in the simultaneous FPA we can not prove a similar proposition to Proposition 2. However, for $c = 1$ we are able to plot the revenue as a function of ε. Figure 7 demonstrates that indeed for some $1 \leq \varepsilon \leq 2$ we have $R^{FPA} < R^{FPA}_{\text{seq}}$

Note that for $\varepsilon \geq 2c$ we have

$$R^{FPA}_{\text{seq}} = R^{FPA}_{sim} = c$$

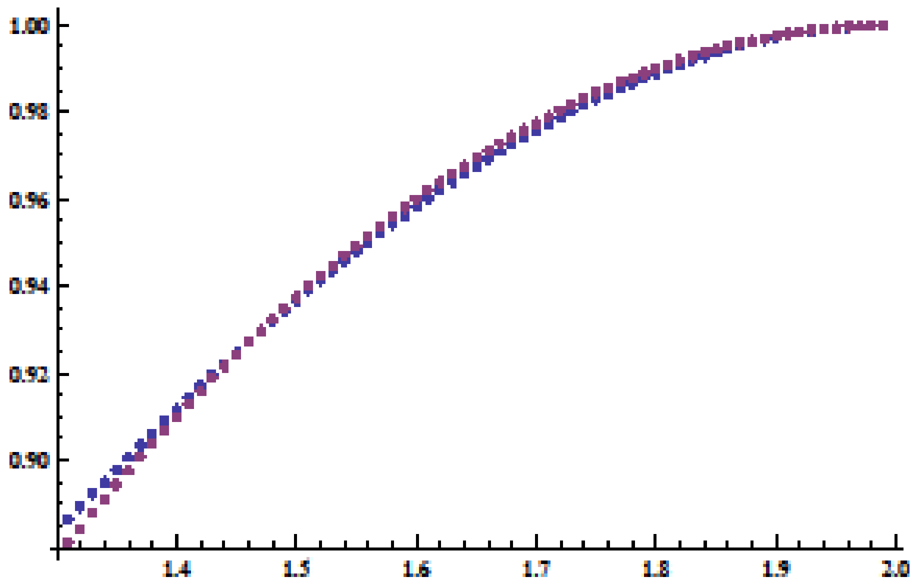

Figure 7: R_{sim}^{FPA} in blue and R_{seq}^{FPA} in purple.

4 More than two bidders

We now present the PBE of the sequential bidding auction for more than two bidders. We focus on the case of stretch of probabilities in order to demonstrate the features of the equilibrium. Assume therefore that F_i is a uniform distribution on $[0, c_i]$. Moreover, order the players by their strength, $c_1 \geq c_2 \geq ... \geq c_n$. We start by analyzing bidder n's behavior. Bidder n observes all previous bids. Denote by $\gamma_j = \max_{1 \leq i \leq j-1} b_i$ the highest bid among the first $j-1$ bids, then bidder n will either bid γ_n, if $v_n \geq \gamma_n$ or otherwise is indifferent between any bid below γ_n. We assume that she bids zero in this case. Therefore we have

$$b_n(v_n; b_1, ..., b_{n-1}) = \begin{cases} 0 & \text{if} \quad 0 \leq v_n < \min\{\gamma_n, c_n\} \\ \gamma_n & \text{if} \quad \min\{\gamma_n, c_n\} \leq v_n \leq c_n \end{cases} \tag{5}$$

Bidder $n-1$ considers both the probability that bidder n will bid zero and the bids of bidders $1, ..., n-1$. Her maximization problem is given by

$$\max\left\{\max_{0 \leq b \leq c_n} \{F_n(b)(v_{n-1} - b)\}, v_{n-1} - c_n\right\} \tag{6}$$

$$\text{s.t.} \quad b \geq \gamma_{n-1}$$

or zero if $F_n(\gamma_{n-1})(v_{n-1} - \gamma_{n-1}) < 0$

Then, if $0 \leq \gamma_{n-1} \leq c_n$ then

$$b_{n-1}\left(v_{n-1}; b_1, ..., b_{n-2}\right) = \begin{cases} 0 & \text{if} \quad 0 \leq v_{n-1} \leq \gamma_{n-1} \\ \gamma_{n-1} & \text{if} \quad \gamma_{n-1} \leq v_{n-1} \leq \min\left\{2\gamma_{n-1}, c_{n-1}\right\} \\ \frac{1}{2}v_{n-1} & \text{if} \quad \min\left\{2\gamma_{n-1}, c_{n-1}\right\} \leq v_{n-1} \leq \min\left\{2c_n, c_{n-1}\right\} \\ c_n & \text{if} \quad \min\left\{2c_n, c_{n-1}\right\} \leq v_{n-1} \leq c_{n-1} \end{cases}$$

while if $c_n \leq \gamma_{n-1} \leq c_{n-1}$ then

$$b_{n-1}\left(v_{n-1}; b_1, ..., b_{n-2}\right) = \begin{cases} 0 & \text{if} \quad 0 \leq v_{n-1} \leq \gamma_{n-1} \\ \gamma_{n-1} & \text{if} \quad \gamma_{n-1} \leq v_{n-1} \leq \min\left\{2\gamma_{n-1}, c_{n-1}\right\} \\ \frac{1}{2}v_{n-1} & \text{if} \quad \min\left\{2\gamma_{n-1}, c_{n-1}\right\} \leq v_{n-1} \leq c_{n-1} \end{cases}$$

The maximization problem of bidder j if $0 \leq \gamma_j \leq c_n$ is given by

$$\max\left\{\max_{\gamma_j \leq b \leq c_n}\left\{\prod_{i=j+1}^{n} F_i\left(b\right)\left(v_j - b\right)\right\}, \max_{c_n \leq b \leq c_{n-1}}\left\{\prod_{i=j+1}^{n-1} F_i\left(b\right)\left(v_j - b\right)\right\}, ..., \max_{c_{j+2} \leq b \leq c_{j+1}}\left\{F_{j+1}\left(b\right)\left(v_j - b\right)\right\}\right\}$$

If $c_n \leq \gamma_j \leq c_{n-1}$ then it is

$$\max\left\{\max_{\gamma_j \leq b \leq c_{n-1}}\left\{\prod_{i=j+1}^{n-1} F_i\left(b\right)\left(v_j - b\right)\right\}, ..., \max_{c_{j+2} \leq b \leq c_{j+1}}\left\{F_{j+1}\left(b\right)\left(v_j - b\right)\right\}\right\}$$

and so on. If $c_j \leq \gamma_j$ then $b_j\left(v_j\right) = 0$.

For example,for $n = 3$, if $c_j = c + (3 - j)\,\varepsilon$ for $1 \leq j \leq 3$ we have

$$b_3(v_3; b_1, b_2) = \begin{cases} 0 & \text{if} \quad 0 \leq v_3 < \min\left\{\gamma_3, c\right\} \\ \gamma_3 & \text{if} \quad \min\left\{\gamma_3, c\right\} \leq v_3 \leq c \end{cases}$$

If $0 \leq b_1 < c$ then

$$b_2\left(v_2; b_1\right) = \begin{cases} 0 & \text{if} \quad 0 \leq v_2 \leq \min\left\{b_1, c + \varepsilon\right\} \\ b_1 & \text{if} \quad \min\left\{b_1, c + \varepsilon\right\} \leq v_2 \leq \min\left\{2b_1, c + \varepsilon\right\} \\ \frac{1}{2}v_2 & \text{if} \quad \min\left\{2b_1, c + \varepsilon\right\} \leq v_2 \leq \min\left\{2c, c + \varepsilon\right\} \\ c & \text{if} \quad \min\left\{2c, c + \varepsilon\right\} \leq v_2 \leq c + \varepsilon \end{cases}$$

while if $c \leq b_1 \leq c + \varepsilon$

$$b_2\left(v_2; b_1\right) = \begin{cases} 0 & \text{if} \quad 0 \leq v_2 \leq \min\left\{b_1, c + \varepsilon\right\} \\ b_1 & \text{if} \quad \min\left\{b_1, c + \varepsilon\right\} \leq v_2 \leq \min\left\{2b_1, c + \varepsilon\right\} \\ \frac{1}{2}v_2 & \text{if} \quad \min\left\{2b_1, c + \varepsilon\right\} \leq v_2 \leq c + \varepsilon \end{cases}$$

and

$$b_1\left(v_1\right) = \begin{cases} \frac{2}{3}v_1 & \text{if} \quad 0 \leq v_1 \leq \min\left\{\frac{3}{2}c, c + 2\varepsilon\right\} \\ c & \text{if} \quad \min\left\{\frac{3}{2}c, c + 2\varepsilon\right\} \leq v_1 \leq \min\left\{2c, c + 2\varepsilon\right\} \\ \frac{1}{2}v_1 & \text{if} \quad \min\left\{2c, c + 2\varepsilon\right\} \leq v_1 \leq c + 2\varepsilon \end{cases}$$

Note that when there are more than two bidders the expected revenue is no longer equal to the expected bid of the first bidder. We have

$$R_{\text{seq}}^{FPA}=\int_0^{c_1}\left(\int_0^{c_2}\left(\int ...\int_0^{c_{n-1}}\left(\max\left\{b_1\left(v_1\right),...,b_{n-1}\left(v_{n-1}\right)\right\}\right)\frac{1}{c_{n-1}}dv_{n-1}...\right)\frac{1}{c_2}dv_2\right)\frac{1}{c_1}dv_1$$

For the example above with three bidders we have, for $0\leq\varepsilon\leq\frac{1}{4}c$

$$\begin{aligned}R_{\text{seq}}^{FPA} &= \int_0^{\frac{3}{4}c+\frac{3}{4}\varepsilon}\left(\int_0^{\frac{4}{3}v_1}\left(\frac{2}{3}v_1\right)\frac{1}{c+\varepsilon}dv_2+\int_{\frac{4}{3}v_1}^{c+\varepsilon}\left(\frac{1}{2}v_2\right)\frac{1}{c+\varepsilon}dv_2\right)\frac{1}{c+2\varepsilon}dv_1\\ &+\int_{\frac{3}{4}c+\frac{3}{4}\varepsilon}^{c+2\varepsilon}\left(\int_0^{c+\varepsilon}\left(\frac{2}{3}v_1\right)\frac{1}{c+\varepsilon}dv_2\right)\frac{1}{c+2\varepsilon}dv_1\end{aligned}$$

and for any ε we have

$$R_{\text{seq}}^{FPA}=\begin{cases}\frac{1}{48}\frac{70c\varepsilon+67\varepsilon^2+19c^2}{c+2\varepsilon} & \text{if} \quad 0\leq\varepsilon\leq\frac{1}{4}c\\ \frac{1}{16}\frac{34c\varepsilon+\varepsilon^2+5c^2}{c+2\varepsilon} & \text{if} \quad \frac{1}{4}c\leq\varepsilon\leq\frac{1}{2}c\\ \frac{1}{16}\frac{18c\varepsilon+17\varepsilon^2+9c^2}{c+2\varepsilon} & \text{if} \quad \frac{1}{2}c\leq\varepsilon\leq c\\ \frac{1}{24}\frac{23\varepsilon^3+51c\varepsilon^2+51c^2\varepsilon+7c^3}{(c+\varepsilon)(c+2\varepsilon)} & \text{if} \quad c\leq\varepsilon\end{cases}$$

4.1 Monotonicity in the number of bidders

We emphasize that determining the optimal order is crucial in the sequential model. As in the sequential all-pay auction in Segev and Sela (2014a) there exists no monotonicity of the revenue in the number of bidders. For example, with two symmetric bidders such that F_1 and F_2 are uniform on $[0,c]$ we have

$$R_{\text{seq}}^{FPA}\left([0,c],[0,c]\right)=\int_0^c\left(\frac{1}{2}v_1\right)\frac{1}{c}dv_1=\frac{1}{4}c$$

If we now add a strong bidder with F_3 uniform on $[\varepsilon,c+\varepsilon]$ such that $0\leq\varepsilon\leq c$ and place her last then we have

$$b_1\left(v_1\right)=\begin{cases}0 & \text{if} \quad 0\leq v_1\leq\varepsilon\\ \frac{1}{3}\left(v_1+\varepsilon+\sqrt{\left(v_1-\varepsilon\right)^2+v_1\varepsilon}\right) & \text{if} \quad \varepsilon\leq v_1\leq c\end{cases}$$

and, for $0\leq v_1\leq\varepsilon$

$$b_2\left(v_2\right)=\begin{cases}0 & \text{if} \quad 0\leq v_2\leq\varepsilon\\ \frac{1}{2}\left(v_2+\varepsilon\right) & \text{if} \quad \varepsilon\leq v_2\leq c\end{cases}$$

for $\varepsilon\leq v_1\leq\frac{(3c-\varepsilon)(c+\varepsilon)}{4c}$

$$b_2\left(v_2\right)=\begin{cases}0 & \text{if} \quad 0\leq v_2\leq\frac{1}{3}\left(v_1+\varepsilon+\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\\ b_1\left(v_1\right) & \text{if} \quad \frac{1}{3}\left(v_1+\varepsilon+\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\leq v_2\leq\frac{1}{3}\left(2v_1-\varepsilon+2\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\\ \frac{1}{2}\left(v_2+\varepsilon\right) & \text{if} \quad \frac{1}{3}\left(2v_1-\varepsilon+2\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\leq v_2\leq c\end{cases}$$

and for $\frac{(3c-\varepsilon)(c+\varepsilon)}{4c}\leq v_1\leq c$

$$b_2\left(v_2\right)=\begin{cases}0 & \text{if} \quad 0\leq v_2\leq\frac{1}{3}\left(v_1+\varepsilon+\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\\ b_1\left(v_1\right) & \text{if} \quad \frac{1}{3}\left(v_1+\varepsilon+\sqrt{\varepsilon^2+v_1^2-\varepsilon v_1}\right)\leq v_2\leq c\end{cases}$$

Therefore

$$
\begin{aligned}
& R_{\text{seq}}^{FPA}\left([0,c],[0,c],[\varepsilon,c+\varepsilon]\right) \\
= \quad & \frac{1}{48}\frac{-42\varepsilon^3+5c\varepsilon^2+22c^2\varepsilon+11c^3}{c^2}-\frac{(\varepsilon-2c)}{12c}\sqrt{(c-\varepsilon)^2+c\varepsilon}+\frac{\varepsilon^2}{8c}\ln\left(\frac{c-\frac12\varepsilon+\sqrt{(c-\varepsilon)^2+c\varepsilon}}{\frac32 c}\right)
\end{aligned}
$$

For example, for $c=1$ we plot in figure 8 both $R_{\text{seq}}^{FPA}([0,c],[0,c])$ in red and $R_{\text{seq}}^{FPA}([0,c],[0,c],[\varepsilon,c+\varepsilon])$ in black as a function of ε and when ε is large enough ($\varepsilon>0.847$) then we have $R_{\text{seq}}^{FPA}([0,c],[0,c],[\varepsilon,c+\varepsilon])<R_{\text{seq}}^{FPA}([0,c],[0,c])$ i.e., adding a third bidder decreased the expected revenue.

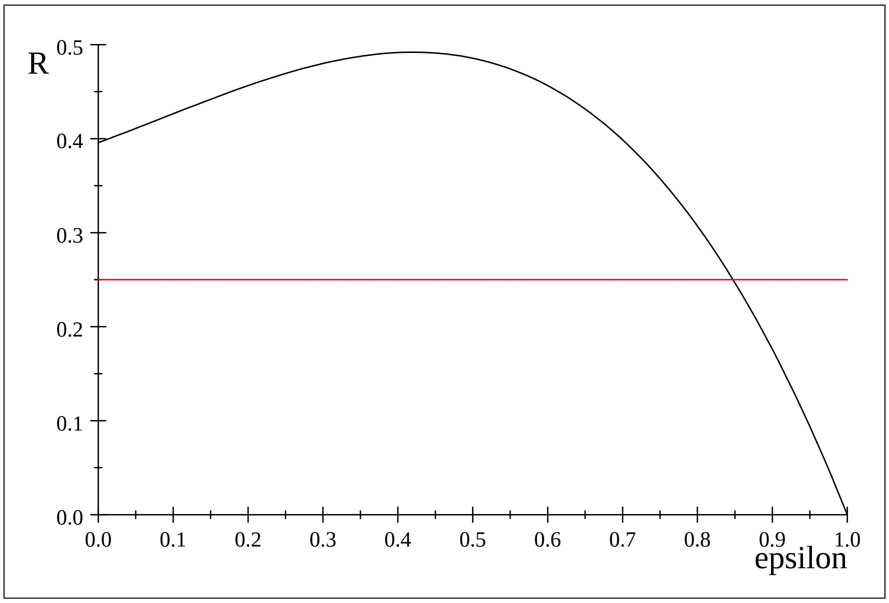

Figure 8

However, if we place the stronger bidder first instead of last then we have

$$
\begin{aligned}
& R_{\text{seq}}^{FPA}\left([\varepsilon,c+\varepsilon],[0,c],[0,c]\right) \\
= \quad & \begin{cases}
\int_\varepsilon^{\frac34 c}\left(\int_0^{\frac43 v_1}\left(\frac23 v_1\right)\frac1c dv_2+\int_{\frac43 v_1}^{c}\left(\frac12 v_2\right)\frac1c dv_2\right)\frac1c dv_1+\int_{\frac34 c}^{c+\varepsilon}\left(\frac23 v_1\right)\frac1c dv_1 \\
\qquad = \frac{1}{432}\frac{-64\varepsilon^3+144c\varepsilon^2+180c^2\varepsilon+171c^3}{c^2} & \text{if} \quad 0\le\varepsilon\le\frac12 c \\
\int_\varepsilon^{\frac34 c}\left(\int_0^{\frac43 v_1}\left(\frac23 v_1\right)\frac1c dv_2+\int_{\frac43 v_1}^{c}\left(\frac12 v_2\right)\frac1c dv_2\right)\frac1c dv_1+\int_{\frac34 c}^{\frac32 c}\left(\frac23 v_1\right)\frac1c dv_1 \\
\qquad +\int_{\frac32 c}^{c+\varepsilon} c\frac1c dv_1=\frac{1}{432}\frac{-64\varepsilon^3+324c^2\varepsilon+135c^3}{c^2} & \text{if} \quad \frac12 c\le\varepsilon\le\frac34 c \\
\int_\varepsilon^{\frac32 c}\left(\frac23 v_1\right)\frac1c dv_1+\int_{\frac32 c}^{c+\varepsilon} c\frac1c dv_1=\frac{1}{12}\frac{12c\varepsilon-4\varepsilon^2+3c^2}{c} & \text{if} \quad \frac34 c\le\varepsilon\le c
\end{cases}
\end{aligned}
$$

which is indeed always higher than $R_{\text{seq}}^{FPA}([0,c],[0,c])$.

5 Concluding remarks

We analyze a sequential bidding first price auction and demonstrate that it may yield a higher expected revenue than both the simultaneous bidding first price auction and the second price auction when the stronger bidder bids first. We therefore argue that when there is reason to believe that bidders are asymmetric the seller may want to consider ordering them and let them bid one by one. By doing that she may not only increase the expected payoff of the weak bidder but also her expected revenue. If bidders are asymmetric enough we show that the sequential FPA may also increase efficiency compared to the simultaneous FPA. This happens however when the stronger bidder bids second.

The lower efficiency achieved in the sequential bidding auction when the stronger bidder bids first is mainly due to the preferred position of the second bidder (when she is the weaker bidder) which allows her to win the auction easily by bidding the same bid as the first bidder. We leave for future research examining mechanisms that could potentially increase the efficiency (when the revenue is already higher than in the simultaneous FPA) by reducing the second bidder's advantage. One such mechanism may be giving a head start to the first bidder. In Segev and Sela (2014b) we discuss head starts in the sequential all-pay auction in which the second bidder can win either by bidding strictly more than the first bidder by some constant (an additive head start) or by bidding the first bidder's bid multiplied by some constant larger than one (multiplicative head start) and show that head starts can increase both efficiency and expected revenue. Other such mechanisms may be a reserve price, a minimal bid or a winning bid mechanisms.

Other important questions such as determining the optimal order of bidders for a general number of bidders and general distributions are also still open and will have to be addressed in continuation research.

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6 Appendix

Proof of Proposition 2

For $\varepsilon = 0, R_{sim} = \frac{1}{3}c > R_{\text{seq}} = \frac{1}{4}c$. Define the difference between the expected revenue in both auctions as

$$
\begin{aligned}
g(\varepsilon) &= R_{sim}^{FPA} - R_{\text{seq}}^{FPA} \\
&= \begin{cases} \frac{c^2(\varepsilon+c)^2}{\varepsilon(\varepsilon+2c)\sqrt{\varepsilon(\varepsilon+2c)}}\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)} + 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) + \frac{1}{4}\frac{(2c-\varepsilon)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{c^2(\varepsilon+c)}\right) & \text{if} \quad 0 \le \varepsilon \le c \\ \frac{c^2(\varepsilon+c)^2}{(\varepsilon+2c)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)} + 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) + \frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3}\right) & \text{if} \quad c \le \varepsilon \end{cases}
\end{aligned}
$$

We first show that $g(\varepsilon) < 0$ for every $\varepsilon \ge c$. Note that when $\varepsilon \ge c$, $g(\varepsilon) = \frac{c^2(\varepsilon+c)^2}{(\varepsilon+2c)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}} f(\varepsilon)$ for

$$
f(\varepsilon) = \ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)} + 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) + \frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3}
$$

therefore we show that $f(\varepsilon) < 0$ for every $\varepsilon \ge c$. Now

$$
f(\varepsilon) < 0 \Leftrightarrow \frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3} < \ln\frac{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)} - 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)
$$

The r.h.s is an increasing function of ε since $\frac{d}{d\varepsilon}\left(\ln\frac{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)} - 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)\right) = \frac{\left((\varepsilon+c)\sqrt{\varepsilon^2+2c\varepsilon}-c(\varepsilon+2c)\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)}{\varepsilon^3+3\varepsilon^2c+2\varepsilon c^2}$ and $(\varepsilon+c)\sqrt{\varepsilon^2+2c\varepsilon} - c(\varepsilon+2c)\sqrt{\frac{\varepsilon}{\varepsilon+2c}} > 0 \Leftrightarrow (\varepsilon+c)\sqrt{\varepsilon^2+2c\varepsilon} > c(\varepsilon+2c)\sqrt{\frac{\varepsilon}{\varepsilon+2c}} \Leftrightarrow (\varepsilon+c)^2(\varepsilon^2+2c\varepsilon) - c^2(\varepsilon+2c)\varepsilon = \varepsilon^2(\varepsilon+2c)^2 > 0$. Moreover $\left(\ln\frac{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)} - 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)\right)|_{\varepsilon=c} = \ln\frac{(3+\sqrt{3})}{(3-\sqrt{3})} - 2\arctan\left(\sqrt{\frac{1}{3}}\right) = 0.269\,76$. Therefore $\ln\frac{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)} - 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) \ge 0.269\,76$ for every $\varepsilon \ge c$. The l.h.s is first increasing in ε and then decreasing, since $\frac{d}{d\varepsilon}\left(\frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3}\right) = -\frac{c\left(\varepsilon^2+\varepsilon c-3c^2\right)}{(\varepsilon+c)^4}\frac{\sqrt{\varepsilon(\varepsilon+2c)}}{\varepsilon+2c}$ and $\varepsilon^2+\varepsilon c-3c^2 > 0 \Leftrightarrow \varepsilon > \frac{1}{2}c\left(\sqrt{13}-1\right) = 1.302\,8c$. Therefore, the maximum of the l.h.s is achieved when $\varepsilon = \frac{1}{2}c\left(\sqrt{13}-1\right)$ and then $\frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3}|_{\varepsilon=\frac{1}{2}c\left(\sqrt{13}-1\right)} = \frac{2\left(\sqrt{13}-1\right)\sqrt{2\sqrt{13}+10}}{\left(1+\sqrt{13}\right)^3} = 0.221\,3$. Thus $\frac{c\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{(\varepsilon+c)^3} \le \frac{2\left(\sqrt{13}-1\right)\sqrt{2\sqrt{13}+10}}{\left(1+\sqrt{13}\right)^3} < \ln\frac{(3+\sqrt{3})}{(3-\sqrt{3})} - 2\arctan\left(\sqrt{\frac{1}{3}}\right) \le \ln\frac{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)} - 2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)$. We now show that $g(\varepsilon)$ is decreasing in ε on $[0,c]$. We have

$$
\frac{d}{d\varepsilon}g(\varepsilon) = -\frac{4c^2(\varepsilon+c)\left(\varepsilon^2+3c^2+2\varepsilon c\right)\sqrt{\varepsilon^2+2c\varepsilon}}{4\varepsilon^3(\varepsilon+2c)^3}\left(\begin{array}{c} \frac{\varepsilon^2(\varepsilon+2c)\left(\varepsilon^3+4c^3+4\varepsilon c^2+4\varepsilon^2 c\right)}{4c^2(\varepsilon+c)(\varepsilon^2+3c^2+2\varepsilon c)\sqrt{\varepsilon^2+2c_1\varepsilon}} \\ +\left(\ln\left(\frac{\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}}{\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}}\right) + 2\arctan\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)\end{array}\right) < 0 \Leftrightarrow
$$

$$
\frac{\varepsilon^2(\varepsilon+2c)\left(\varepsilon^3+4c^3+4\varepsilon c^2+4\varepsilon^2 c\right)}{4c^2(\varepsilon+c)(\varepsilon^2+3c^2+2\varepsilon c)\sqrt{\varepsilon^2+2c\varepsilon}} > \ln\left(\frac{\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}}{\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}}\right) - 2\arctan\sqrt{\frac{\varepsilon}{\varepsilon+2c}}
$$

Both the r.h.s and the l.h.s of this inequality are increasing functions of ε on $[0,c]$ since $\frac{d}{d\varepsilon}\left(\frac{\varepsilon^2(\varepsilon+2c)\left(\varepsilon^3+4c^3+4\varepsilon c^2+4\varepsilon^2 c\right)}{4c^2(\varepsilon+c)(\varepsilon^2+3c^2+2\varepsilon c)\sqrt{\varepsilon^2+2c\varepsilon}}\right) = \frac{1}{4}\varepsilon^2(\varepsilon+2c)\frac{\left(2\varepsilon^7+36c^7+188\varepsilon^2c^5+215\varepsilon^3c^4+150\varepsilon^4c^3+63\varepsilon^5c^2+104\varepsilon c^6+16\varepsilon^6 c\right)}{c^2\left(\sqrt{\varepsilon^2+2\varepsilon c}\right)^3(\varepsilon+c)^2(\varepsilon^2+3c^2+2\varepsilon c)^2} > 0$ and $\frac{d}{d\varepsilon}\left(\ln\left(\frac{\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}}{\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}}\right) - 2\arctan\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) = \frac{\varepsilon}{(\varepsilon+c)\sqrt{\varepsilon^2+2c\varepsilon}} > 0$. Moreover, $\lim_{\varepsilon\to 0}\left(\frac{\varepsilon^2(\varepsilon+2c)\left(\varepsilon^3+4c^3+4\varepsilon c^2+4\varepsilon^2 c\right)}{4c^2(\varepsilon+c)(\varepsilon^2+3c^2+2\varepsilon c)\sqrt{\varepsilon^2+2c\varepsilon}}\right) = 0$ and $\lim_{\varepsilon\to 0}\left(\ln\left(\frac{\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}}{\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}}\right) - 2\arctan\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right) = 0$.

Finally, the difference between the l.h.s and the r.h.s is increasing in ε on $[0,c]$ since

$$\frac{d}{d\varepsilon}\left(\frac{\varepsilon^2(\varepsilon+2c)\left(\varepsilon^3+4c^3+4\varepsilon c^2+4\varepsilon^2 c\right)}{4c^2(\varepsilon+c)\left(\varepsilon^2+3c^2+2\varepsilon c\right)\sqrt{\varepsilon^2+2c\varepsilon}}-\left(\ln\left(\frac{\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}}{\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}}\right)-2\arctan\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)\right)$$

$$= \frac{\varepsilon^3(\varepsilon+2c)\left(2\varepsilon^6+20c^6+159\varepsilon^2c^4+130\varepsilon^3c^3+59\varepsilon^4c^2+100\varepsilon c^5+16\varepsilon^5 c\right)}{4\left(\sqrt{\varepsilon^2+2\varepsilon c}\right)^3 c^2(\varepsilon+c)^2\left(\varepsilon^2+3c^2+2\varepsilon c\right)^2} > 0.$$

Moreover, since

$g(0) = \lim_{\varepsilon\to 0}\left(\frac{c^2(\varepsilon+c)^2}{\varepsilon(\varepsilon+2c)\sqrt{\varepsilon(\varepsilon+2c)}}\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon^2+2c\varepsilon}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon^2+2c\varepsilon}\right)}+2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)+\frac{1}{4}\frac{(2c-\varepsilon)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{c^2(\varepsilon+c)}\right)\right) = \frac{1}{12}c > 0$

and

$g(c) = \frac{4c}{3\sqrt{3}}\left(\ln\frac{\left(3-\sqrt{3}\right)}{\left(3+\sqrt{3}\right)}+2\arctan\left(\sqrt{\frac{1}{3}}\right)+\frac{\sqrt{3}}{8}\right) = -0.04099\,5c < 0$ we conclude that there exists a cut-off $0<\varepsilon^*(c)\le c$ such that for all $\varepsilon<\varepsilon^*(c), R_{sim}^{FPA} > R_{\text{seq}}^{FPA}$ and for all $\varepsilon>\varepsilon^*(c), R_{sim}^{FPA} < R_{\text{seq}}^{FPA}$. It remains to show that $\varepsilon^*(c)$ is increasing in c. The cutoff $\varepsilon^*(c)$ is defined by the following equation

$$g(\varepsilon)=0 \Leftrightarrow \ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}+2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)+\frac{1}{4}\frac{(2c-\varepsilon)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{c^2(\varepsilon+c)}=0$$

Then $\frac{d\varepsilon^*(c)}{dc} = -\frac{\frac{d}{dc}\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}+2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)+\frac{1}{4}\frac{(2c-\varepsilon)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{c^2(\varepsilon+c)}\right)}{\frac{d}{d\varepsilon}\left(\ln\frac{\left(\varepsilon+2c-\sqrt{\varepsilon(\varepsilon+2c)}\right)}{\left(\varepsilon+2c+\sqrt{\varepsilon(\varepsilon+2c)}\right)}+2\arctan\left(\sqrt{\frac{\varepsilon}{\varepsilon+2c}}\right)+\frac{1}{4}\frac{(2c-\varepsilon)\varepsilon\sqrt{\varepsilon(\varepsilon+2c)}}{c^2(\varepsilon+c)}\right)} = \frac{\varepsilon}{c} > 0.$ ■

Proof of Proposition 3

We have

$$R_{sim}^{SPA}-R_{\text{seq}}^{FPA} = \begin{cases} \frac{1}{6}c\frac{3\varepsilon+2c}{\varepsilon+c}-\frac{1}{4}(c+\varepsilon) = -\frac{1}{12}\frac{3\varepsilon^2-c^2}{\varepsilon+c} & \text{if} \quad 0\le\varepsilon\le c \\ \frac{1}{6}c\frac{3\varepsilon+2c}{\varepsilon+c}-\frac{\varepsilon c}{\varepsilon+c} = -\frac{1}{6}c\frac{3\varepsilon-2c}{\varepsilon+c} & \text{if} \quad c\le\varepsilon \end{cases}$$

Therefore it is easy to check that $\varepsilon^{**}(c)=\frac{1}{\sqrt{3}}c<c$. Moreover $\varepsilon^{**}(c)$ is increasing in c. Finally, $R_{sim}^{FPA}(\varepsilon^{**}) - R_{\text{seq}}^{FPA}(\varepsilon^{**}) = \frac{c\left(1+\sqrt{3}\right)^2}{\left(1+2\sqrt{3}\right)\sqrt{\left(\frac{1+2\sqrt{3}}{3}\right)}}\left(\ln\frac{\left(1+2\sqrt{3}-\sqrt{1+2\sqrt{3}}\right)}{\left(1+2\sqrt{3}+\sqrt{1+2\sqrt{3}}\right)}+2\arctan\left(\sqrt{\frac{1}{1+2\sqrt{3}}}\right)+\frac{1}{12}\frac{\left(2\sqrt{3}-1\right)\sqrt{1+2\sqrt{3}}}{\left(1+\sqrt{3}\right)}\right) = 0.01\,959\,7c > 0$ therefore $\varepsilon^{**}(c)<\varepsilon^*(c)$. ■

Proof of Proposition 4

We first need to prove that

$$U_1^{\text{seq}\,FPA} < U_1^{simFPA} < U_1^{SPA}$$

where

$$U_1^{\text{seq}\,FPA} = \begin{cases} \frac{1}{12}\frac{(c+\varepsilon)^2}{c} & \text{if} \quad 0\le\varepsilon\le c \\ \frac{1}{6}\frac{3\varepsilon^2+c^2}{c+\varepsilon} & \text{if} \quad c\le\varepsilon \end{cases}$$

$$U_1^{simFPA} = \frac{(c+\varepsilon)^2}{2\varepsilon(2c+\varepsilon)}\left(\frac{c^2}{\sqrt{\varepsilon(2c+\varepsilon)}}\left(3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)\right)-(c-\varepsilon)\right)$$

$$U_1^{SPA} = \frac{1}{6}\frac{3c\varepsilon+3\varepsilon^2+c^2}{c+\varepsilon}$$

Now $U_1^{simFPA} < U_1^{SPA} \Leftrightarrow g(\varepsilon) = \frac{1}{3}\frac{\left(3\varepsilon^3+7c\varepsilon^2+8c^2\varepsilon+3c^3\right)\sqrt{\varepsilon(2c+\varepsilon)}}{c(c+\varepsilon)^3} - 3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)+2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right) > 0$. Now

$\frac{dg(\varepsilon)}{d\varepsilon}=\frac{d}{d\varepsilon}\left(\frac{1}{3}\frac{\left(3\varepsilon^3+7c\varepsilon^2+8c^2\varepsilon+3c^3\right)\sqrt{\varepsilon(2c+\varepsilon)}}{c(c+\varepsilon)^3}-3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)+2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)\right)=\frac{\varepsilon^2\left(\varepsilon^3+4c\varepsilon^2+5c^2\varepsilon+c^3\right)}{c(c+\varepsilon)^4\sqrt{\varepsilon(2c+\varepsilon)}}>0$ and for $\varepsilon=0,g(0)=0$. Therefore we conclude that indeed $g(\varepsilon)>0$ for every ε.

Moreover, for $0\leq\varepsilon\leq c$, $U_1^{\mathrm{seq}\,FPA}<U_1^{simFPA}\Leftrightarrow f(\varepsilon)=3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)-\frac{1}{6}\frac{\left(-4c\varepsilon+\varepsilon^2+6c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{c^3}>0$. Now

$\frac{df(\varepsilon)}{d\varepsilon}=\frac{d}{d\varepsilon}\left(3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)-\frac{1}{6}\frac{\left(-4c\varepsilon+\varepsilon^2+6c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{c^3}\right)=\frac{\varepsilon(c+\varepsilon)(2c-\varepsilon)}{2c^3\sqrt{\varepsilon(2c+\varepsilon)}}>0$ and for $\varepsilon=0,f(0)=0$. Therefore we conclude that indeed $f(\varepsilon)>0$ for every $0\leq\varepsilon\leq c$.

For $c\leq\varepsilon$, $U_1^{\mathrm{seq}\,FPA}<U_1^{simFPA}\Leftrightarrow h(\varepsilon)=3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)-\frac{1}{3}\frac{\left(8c\varepsilon+\varepsilon^2+3c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)^3}>0$. Now

$\frac{dh(\varepsilon)}{d\varepsilon}=\frac{d}{d\varepsilon}\left(3\ln\left(\frac{c+\varepsilon+\sqrt{\varepsilon(2c+\varepsilon)}}{c}\right)-2\operatorname{arccosh}\left(\frac{c+\varepsilon}{c}\right)-\frac{1}{3}\frac{\left(8c\varepsilon+\varepsilon^2+3c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)^3}\right)=\frac{\varepsilon^2(3c+\varepsilon)^2}{(c+\varepsilon)^4\sqrt{\varepsilon(2c+\varepsilon)}}>0$ and for $\varepsilon=c,h(c)=3\ln\left(2+\sqrt{3}\right)-2\operatorname{arccosh}(2)-\frac{\sqrt{3}}{2}=0.450\,93>0$. Therefore we conclude that indeed $h(\varepsilon)>0$ for every $c\leq\varepsilon$.

Next we wish to prove that

$$U_2^{SPA}<U_2^{simFPA}<U_2^{\mathrm{seq}\,FPA}$$

where

$$\begin{aligned}
U_2^{SPA} &= \frac{1}{6}\frac{c^2}{c+\varepsilon}\\
U_2^{simFPA} &= \frac{c^2(c+\varepsilon)^2}{2\varepsilon(2c+\varepsilon)\sqrt{\varepsilon(2c+\varepsilon)}}\left(\frac{1}{2}\pi-\arcsin\left(\frac{c}{c+\varepsilon}\right)-2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)\right)+\frac{1}{2}\frac{c^2(c+2\varepsilon)}{\varepsilon(2c+\varepsilon)}\\
U_2^{\mathrm{seq}\,FPA} &= \begin{cases}\frac{1}{24}\frac{7c^2-4c\varepsilon+\varepsilon^2}{c} & \text{if} \quad 0\leq\varepsilon\leq c\\ \frac{1}{3}\frac{c^2}{c+\varepsilon} & \text{if} \quad c\leq\varepsilon\end{cases}
\end{aligned}$$

First we show that $U_2^{SPA}<U_2^{simFPA}$. We have $U_2^{SPA}<U_2^{simFPA}\Leftrightarrow g(\varepsilon)=\frac{1}{2}\pi-\arcsin\left(\frac{c}{c+\varepsilon}\right)-2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)+\frac{1}{3}\frac{\left(7c\varepsilon+5\varepsilon^2+3c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)^3}>0$. Now

$\frac{dg(\varepsilon)}{d\varepsilon}=\frac{d}{d\varepsilon}\left(\frac{1}{2}\pi-\arcsin\left(\frac{c}{c+\varepsilon}\right)-2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)+\frac{1}{3}\frac{\left(7c\varepsilon+5\varepsilon^2+3c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)^3}\right)=\frac{c^2\varepsilon^2}{(c+\varepsilon)^4\sqrt{\varepsilon(2c+\varepsilon)}}>0$ and for $\varepsilon=0,g(0)=0$. Therefore we conclude that indeed $g(\varepsilon)>0$ for every ε.

Moreover, for $0\leq\varepsilon\leq c$ we have $U_2^{simFPA}<U_2^{\mathrm{seq}\,FPA}\Leftrightarrow f(\varepsilon)=\frac{1}{12}\frac{\left(\varepsilon^3-3c\varepsilon^2+2c^2\varepsilon-12c^3\right)\sqrt{\varepsilon(2c+\varepsilon)}}{c^3(c+\varepsilon)}-\frac{1}{2}\pi+\arcsin\left(\frac{c}{c+\varepsilon}\right)+2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)>0$. Now

$\frac{df(\varepsilon)}{d\varepsilon}=\frac{\varepsilon\left(\varepsilon^3(c+\varepsilon)+3c^2(c-\varepsilon)(2c+\varepsilon)\right)}{4c^3(c+\varepsilon)^2\sqrt{2c\varepsilon+\varepsilon^2}}>0$ and for $\varepsilon=0,f(0)=0$. Therefore we conclude that indeed $f(\varepsilon)>0$ for every $0\leq\varepsilon\leq c$.

Finally, for $c\leq\varepsilon$ we have $U_2^{simFPA}<U_2^{\mathrm{seq}\,FPA}\Leftrightarrow h(\varepsilon)=-\frac{1}{3}\frac{\left(5c\varepsilon+4\varepsilon^2+3c^2\right)\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)^3}-\frac{1}{2}\pi+\arcsin\left(\frac{c}{c+\varepsilon}\right)+2\left(\arcsin\frac{\sqrt{\varepsilon(2c+\varepsilon)}}{(c+\varepsilon)}\right)>0$. Now

$\frac{dh(\varepsilon)}{d\varepsilon}=\frac{2c^3\varepsilon}{(c+\varepsilon)^4\sqrt{\varepsilon(2c+\varepsilon)}}>0$ and for $\varepsilon=c,h(c)=-\frac{\sqrt{3}}{2}-\frac{1}{2}\pi+\arcsin\left(\frac{1}{2}\right)+2\left(\arcsin\frac{\sqrt{3}}{2}\right)=0.181\,17>0$. Therefore we conclude that indeed $h(\varepsilon)>0$ for every $c\leq\varepsilon$. ■

Proof of Proposition 5

We have

$$R_{sim}^{SPA} - R_{\text{seq}}^{FPA} = \begin{cases} \frac{1}{12}\frac{2\varepsilon^3 - 6c\varepsilon^2 + c^3}{c^2} & \text{if} \quad 0 \leq \varepsilon \leq c \\ -\frac{1}{4}\frac{\left(2c^2 - (2c-\varepsilon)^2\right)}{c} & \text{if} \quad c \leq \varepsilon \leq 2c \\ -\frac{1}{2}c & \text{if} \quad 2c \leq \varepsilon \end{cases}$$

For $\varepsilon = 0$ we have $R_{sim}^{SPA} - R_{\text{seq}}^{FPA} = \frac{1}{12}c$. When $0 \leq \varepsilon \leq c$, we have $R_{sim}^{SPA} - R_{\text{seq}}^{FPA} = 0 \Leftrightarrow 2\varepsilon^3 - 6c\varepsilon^2 + c^3 = 0$, and the solution is:

$$\varepsilon^{**}(c) = \left(1 - \frac{1}{2}\frac{1}{\sqrt[3]{\sqrt{-\frac{7}{16}} + \frac{3}{4}}} + \frac{1}{2}i\sqrt{3}\left(\frac{1}{\sqrt[3]{\sqrt{-\frac{7}{16}} + \frac{3}{4}}} - \sqrt[3]{\sqrt{-\frac{7}{16}} + \frac{3}{4}}\right) - \frac{1}{2}\sqrt[3]{\sqrt{-\frac{7}{16}} + \frac{3}{4}}\right)c = 0.442\,13c$$

Moreover $\frac{d}{d\varepsilon}\left(R_{sim}^{SPA} - R_{\text{seq}}^{FPA}\right) = \frac{d}{d\varepsilon}\left(\frac{1}{12}\frac{2\varepsilon^3 - 6c\varepsilon^2 + c^3}{c^2}\right) = -\frac{1}{2}\varepsilon\frac{2c - \varepsilon}{c^2} < 0$. Therefore, for all $\varepsilon < \varepsilon^{**}(c)$, $R_{sim}^{SPA} > R_{\text{seq}}^{FPA}$ and for all $\varepsilon > \varepsilon^{**}(c)$, $R_{sim}^{SPA} < R_{\text{seq}}^{FPA}$. Finally when $c \leq \varepsilon \leq 2c$ and when $2c \leq \varepsilon$ we have $R_{sim}^{SPA} - R_{\text{seq}}^{FPA} < 0$. ■