Eliminations of Dominated Strategies and Inessential Players: An Abstraction Process

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Abstraction of a social situation
- Choices of relevant factors
- Eliminations of irrelevant factors **(negative criterion)**

⇨ **A Theory (description + positive criterion)**

Consider iterated eliminations of **dominated strategies** and **inessential players**

negative criterion – a target should not be chosen

(Positive criterion – a target is recommended to be chosen)

Main questions:

(1) **Preservation** of some essential properties (positive criteria) in such an abstraction process?

(2) **Lengths** of generated sequences and **order independence?**

(3) **Possible shapes** of generated sequences?

Example 1: Battle of the Sexes with the 2nd Boy

Assumption: When a boy is not chosen, his payoff depends upon whether dating of the others is successful or not.

- **Players**: Boy 1, 2, and Girl.
- **Strategies**:
  - Boy 1: s_{11} (boxing 1), s_{12} (cinema 1)
  - ~~Boy 2: s_{21} (boxing 2), s_{22} (cinema 2)~~
  - Girl : s_{31} (boxing 1), s_{32} (cinema 1)

    ~~s_{33} (boxing 2), s_{34} (cinema 2)~~.

1/3 (2)	s_{31}	s_{32}
s_{11}	15, 10 (-10)	5,5 (-5)
s_{12}	5,5 (-5)	10, 15 (-10)

2/3 (1)	s_{33}	s_{34}
s_{21}	15, 1 (-10)	5,0 (-5)
s_{22}	5,0 (-5)	10, 2 (-10)

Step 1: Elim. of ds $\rightarrow$ **Step 2: Elim. of ip**

⇓

Two-person battle of the sexes

Dominated Strategies and Inessential Players

- $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$: a ***finite strategic form game***
  - N: a finite ***set of players***;
  - S_i: a **nonempty** finite ***set of strategies*** for i;
  - h_i: $\Pi_{j \in N} S_j \to \mathbb{R}$: a ***payoff function*** for i.
  - ➢ When N is empty, the game is the ***empty game***, denoted as $G_\emptyset$.

- Let $s_i, s_i' \in S_i$. We say that ***s'_i dominates s_i*** iff

$$h_i(s_i'; s_{-i}) > h_i(s_i; s_{-i}) \; for \; all \; s_{-i} \in S_{-i}.$$

- We say that $i \in N$ is an ***inessential player*** in G iff for all $j \in N$

$$h_j(s_i; s_{-i}) = h_j(s_i'; s_{-i}) \; for \; all \; s_i, s_i' \in S_i \; and \; s_{-i} \in S_{-i}.$$

Example 2.

1 \ 2	s_{21}	s_{22}
s_{11}	5, 1	5, 1
s_{12}	6, 3	6, 3

Player 2 is an inessential player, but player 1 is not.

"Inessential player" and "dummy player" in cooperative game theory are logically independent.

- Player i is *dummy* iff $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N - \{i\}$.

Example 3 (Inessential $\not\Rightarrow$ Dummy)

$1 \setminus 2$	s_{21}	s_{22}
s_{11}	0,1	1,1
s_{12}	0,1	1,1

- 1 is inessential in G.
- The maxmin cooperative game (N, v) from G :

$\mathrm{v}(\{1\}) = 0, v(\{2\}) = 1, v(\{1,2\}) = 2.$

- 1 is **not** a dummy player in (N, v).

Example 4 (Dummy $\not\Rightarrow$ Inessential)

$1 \setminus 2$	s_{21}	s_{22}
s_{11}	1,4	3,2
s_{12}	1,4	2,3

- The maxmin cooperative game (N, v) from G :

$\mathrm{v}(\{1\}) = 1, v(\{2\}) = 4, v(\{1,2\}) = 5.$

- Both 1 and 2 are dummy in (N, v).
- **Neither** 1 **nor** 2 is inessential in G.

On the other hand, we have

Let G be an n-person game. If all players are inessential, then all players are dummy players in the derived cooperative game.

Lemma 1. Let I be a set of inessential players. Then for all $j \in N$,
$$h_j(s_I; s_{-I}) = h_j(s_I'; s_{-I}) \text{ for all } s_I, s_I' \in S_I \text{ and } s_{-I} \in S_{-I}.$$

We restrict the payoff function h_i to the domain $\Pi_{j \in N-I} S_j$, denoted by $h_i{}'$:
$$h_i'(s_{N-I}) = h_i(s_{N-I}; s_I) \text{ for all } s_{N-I} \in S_{N-I} \text{ and } s_I \in S_I.$$

Let $G = (N, \{S_i\}_{i \in N}, \{h_i\}_{i \in N})$, $G' = (N', \{S'{}_i\}_{i \in N'}, \{h'{}_i\}_{i \in N'})$.
We say that G' is a ***D-reduction*** of G iff

DR1: $N' \subseteq N$; and any $i \in N - N'$ is inessential in G;
DR2: for all $i \in N', S_i' \subseteq S_i$; and any $s_i \in S_i - S_i{}'$ is dominated in G;
DR3: for all $i \in N', h_i{}'$ is the restriction of h_i to $\Pi_{j \in N'} S_j'$.

Preservation of Nash Equilibrium

Theorem 1. Let G' be a D-reduction of G. Then
(1) if s_N is an NE in G, then its restriction $s_{N'}$ is an NE in G';
(2) if $s_{N'}$ is an NE in G', $(s_{N'}; s_{N-N'})$ is an NE in G for all $s_{N-N'} \in S_{N-N'}$.

This is an extension of a theorem in Maschler et al (2012).

D-reduction

$N' = N$ in DR1 : ***ds*-reduction** $G \to_{ds} G'$

$S_i = S_i'$ for $i \in N'$ in DR2: ***ip*-reduction** $G \to_{ip} G'$

- We say that G'' is a ***DI* - compound reduction** of G iff there is an interpolating G' such that $G \to_{ds} G'$ and $G' \to_{ip} G''$.

$$\underbrace{G \to_{ds} G' \to_{ip} G''}_{DI\text{-compound reduction}}$$

***ID* - compound reduction?**
$G \to_{ip} G' \to_{ds} G''$

ID

$G \to_{ip} G' \to_{ds} G''$

A ds here is also a ds in G.

DI

$G \to_{ds} G' \to_{ip} G''$

An ip here may **not** be an ip in G.

Example 5

1 \ 2	s_{21}	s_{22}
s_{11}	4, 6	2, 7
s_{12}	4, 6	2, 7

$\to_{ds}$

1 \ 2	s_{22}
s_{11}	2, 7
s_{12}	2, 7

$\to_{ip}$ $G_\emptyset$

PL 2 is **not** an ip in G

Hence, DI is more effective.

The IEDI Process and Generated Sequences

- $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell \rangle$ is a ***W-IEDI sequence from $G = G^0$*** iff

(1) for each $t = 0, \dots, \ell - 1$, G^{t+1} is a DI-compound reduction of G^t and $G^t \neq G^{t+1}$;

(2) G^ℓ has no dominated strategies and no inessential players.

Let $\mathbb{W}(G)$ be the set of all W-IEDI from G.

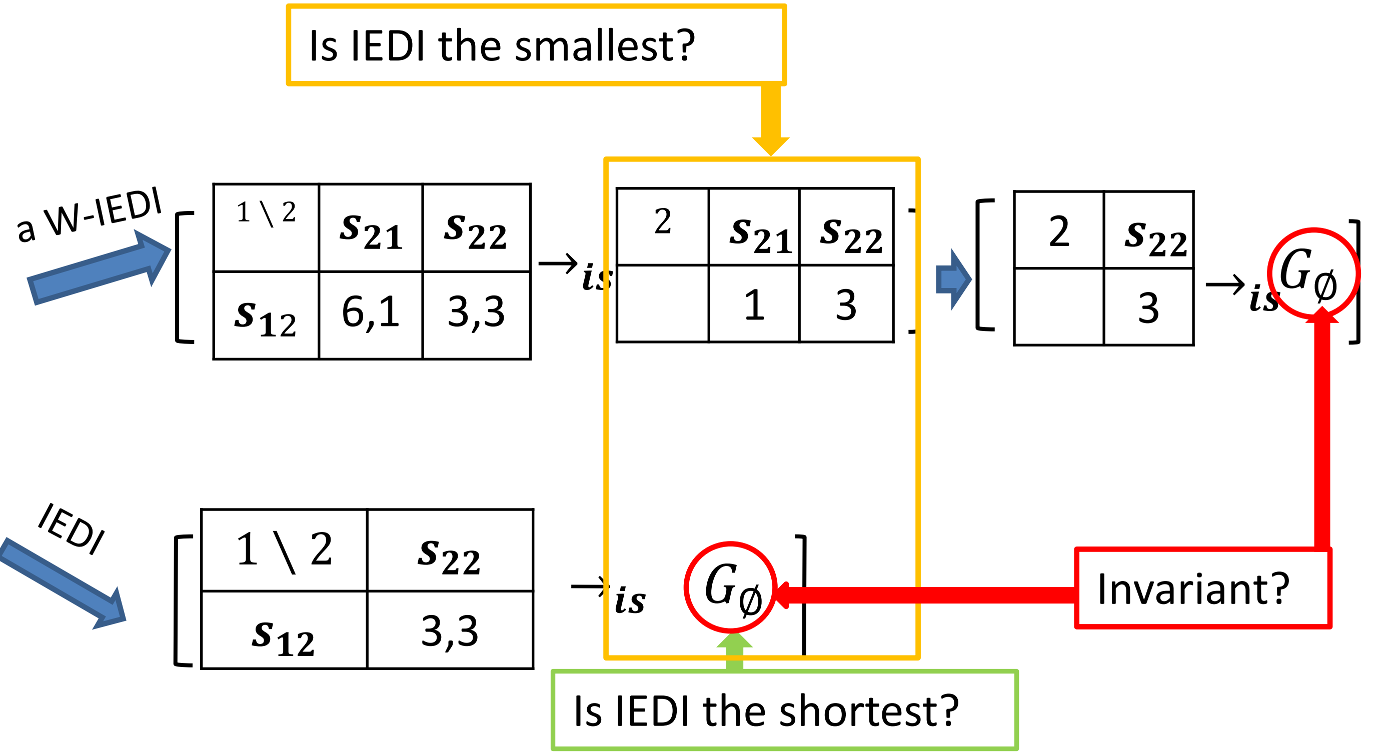
- A WIEDI $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell \rangle$ is said to be the ***IEDI sequence*** iff G^{t+1} is the **strict** DI-compound reduction of G^t for all $t = 0, \dots, \ell - 1$.

- Let G' be a *ds*-reduction of G. When $S_i - S_i' = \{s_i : s_i$ is a dominated strategy for i in $G\}$ in DR2, G' is the ***strict ds-reduction*** of G.
- Let G' be an *ip*-reduction of G.When $N - N' = \{i : i$ is an inessential player in $G\}$ in DR1, G' is the ***strict ip-reduction*** of G.

By Theorem 1, the IEDI process preserves Nash equilibrium.

Example 6

$1 \setminus 2$	s_{21}	s_{22}
s_{11}	5,5	1,6
s_{12}	6,1	3,3



Theorem 2 (The IEDI: Shortest and Smallest). Let G be a finite strategic form game, and $\Gamma^*(G) = \langle G^{*0}, G^{*1}, \dots, G^{*\ell^*} \rangle$ be the IEDI from G. Then, for any W-IEDI $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell \rangle$ from G,

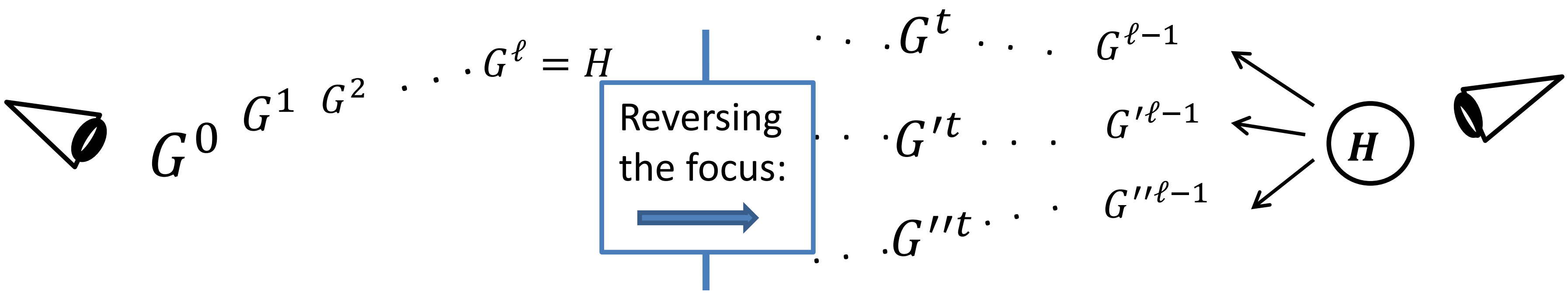
(1) $\ell^* \leq \ell$;

(2) for each $t \leq \ell^*$, G^{*t} is a subgame of G^t;

(3) $G^{*\ell^*}$=G^ℓ.

This is an extension of the order independence theorem (cf. Apt (2011)).

Possible Shapes of IEDI



Key concept: player – configuration (pc)

IEDI $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$

Its player-configuration $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ satisfies PC0 – PC3

IEDI $\Gamma(G)$ exists such that with the player - configuration $[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$

$[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$ satisfies **PC0 - PC3**

Let $\Gamma(G) = \langle G^0, G^1, \ldots, G^\ell \rangle$ be an IEDI. For $t = 0, \ldots, \ell$, we define

$T^t = \{i \in N^t$: player i has a dominated strategy in $G^t\}$;

$[(N^0, T^0), \ldots, (N^\ell, T^\ell)]$: the ***player-configuration*** of $\Gamma(G)$.

Lemma 2 ( ). Let $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell \rangle$ be an IEDI and let $[(N^0, T^0), \dots, (N^\ell, T^\ell)]$ be the player-configuration of $\Gamma(G)$. Then

PC0: $T^t \subseteq N^t$ for each $t = 0, \dots, \ell$;

PC1: there is some m_0 with $0 \leq m_0 \leq \ell$ such that

$$N^0 \supseteq \cdots \supseteq N^{m_0} \supsetneq \cdots \supsetneq N^\ell \; with \; |N^\ell| \neq 1;$$

PC2: $T^{m_0} = \cdots = T^\ell = \emptyset$;

PC3: for $t = 1, \dots, m_0$, if $|T^{t-1}| = 1$, then $T^{t-1} \cap T^t = \emptyset$.

Theorem 3 ( ). Let $H = (N^H, \{S_i{}^H\}_{i \in N^H}, \{h_i{}^H\}_{i \in N^H})$ be a game with no dominated strategies and no inessential players. Let $[(N^0, T^0), \dots, (N^\ell, T^\ell)]$ be any sequence satisfying PC0 – PC3. Then there exists an IEDI $\Gamma(G) = \langle G^0, G^1, \dots, G^\ell \rangle$ such that

(a) $G^\ell = H$;

(b) $[(N^0, T^0), \dots, (N^\ell, T^\ell)]$ is the player configuration of $\Gamma(G)$.

Conclusions

- We presented the following three main theorems.
- **Theorem 1.** NE (among others) is preserved in the IEDI process.

  Thus, this process does not affect the NE solution theory.
- **Theorem 2.** The IEDI is the shortest, smallest (also order independence).

  Thus, the IEDI can be adopted as the representative for a further analysis.
- **Theorem 3.** For any player-configuration (pc) satisfying PC0 – PC3, there is an IEDI having this pc.

  This characterizes the possible IEDI sequences with respect to pc's.

References

Apt, K. R., (2011), "Direct Proofs of Order Independence", *Economics Bulletin* 31, 106 – 115.

Maschler, M., Solan, E., and Zamir, S., (2013), *Game Theory*, Cambridge University Press, Cambridge.