Approximate Quasi-Linearity for Large Incomes

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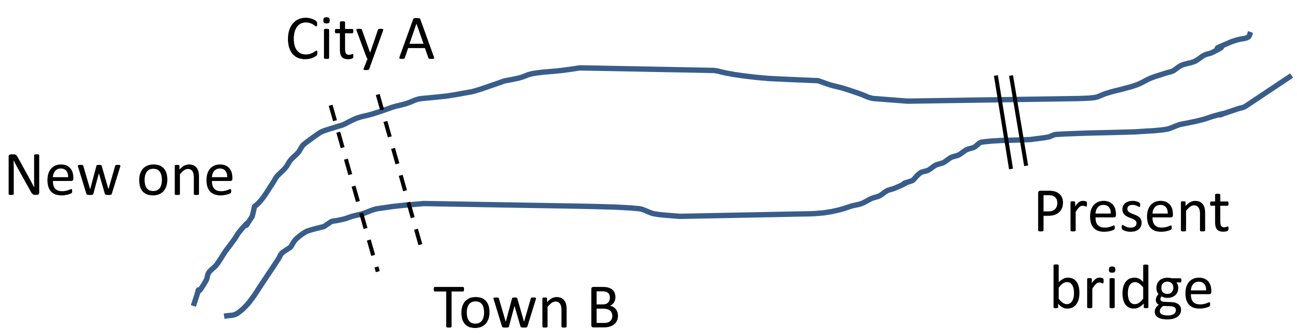
The **quasi-linearity (QL) assumption:**

- A utility function $U_i: X \times R_+ \to R$ is *quasi-linear* iff there is a function $u_i: X \to R$ such that
$$U_i(x, c) = u_i(x) + c \text{ for all } (x, c) \in X \times R_+,$$
where X is an arbitrary nonempty set.

- It is assumed as the first-order approximation:
  - Cost-benefit analysis
  - Cooperative game theory (TU theory)
    - The total benefit for a group of people is expressed by a **single value**.
  - Other economic problems such as auction theory.

Cost-benefit analysis

- Cost-benefit analysis, e.g., to build a bridge (or not);
  - $\sum_{i \in N} u_i(b) - C(b)$ > (or ≤) $\sum_{i \in N} u_i(0) - C(0)$.



- How much can each person in Town B pay for this bridge?
  - $U_i(0, I_i - u_i(0)) = U_i(b, I_i - u_i(b))$ for $i \in N$
  - The "No income effect" assumption: It means $u_i(b) - u_i(0)$'s independent of income I_i.
  - These monetary "willingness to pays" can be summed up over the people.

- The basic assumptions
  - Partial economic world: Money represents the purchasing power for the commodities outside the world in question.
    - **Money is the composite commodity.**
  - No income effects: Changes in incomes do not affect the evaluations of the target economic activities.

- **When is the QL assumption appropriate?**
  - A typical answer is: an individual income is large enough.
  - Is this always true?
  - **E.g., does any utility function exhibits quasi-linearity when income is large enough?**

A preference relation $\precsim$ in terms of a QL function

Let $\precsim_i$ be a preference relation over $X \times R_+$.

- C0: $\precsim_i$ is a complete preordering.
- C1(Monotonicity): if $c < c'$, then $(x, c) \prec_i (x, c')$.
- C2(Substitutability): if $(x, c) \prec_i (x', c')$, then there is an $\alpha > 0$ such that $(x, c + \alpha) \sim_i (x', c')$.
- **C3^{NI}(No income effects): if $(x, c) \sim_i (x', c')$ and $\alpha > 0$, then $(x, c + \alpha) \sim_i (x', c' + \alpha)$.**

- Kaneko (1976) and Kaneko-Wooders (2004).

Theorem: $\precsim_i$ satisfies C0-C2 and $\boldsymbol{C3^{NI}}$ $\Leftrightarrow$ there is a function $u_i: X \to R$ such that for all $(x, c), (x', c') \in X \times R_+$,

$$(x, c) \precsim_i (x', c') \Leftrightarrow u_i(x) + c \leq u_i(x') + c'.$$

Some remarks

- When people in a situation have quasi-linear utility functions, “transfers” of money look like “transfers of utility”.
- This motivates the term “transferable utility”.
- Also, the single value

$$\max_{x} \sum_{i \in N} u_i(x) - C(x)$$

captures the Pareto optimal outcomes.

See Kaneko-Wooders (2004).

C3 (Approximate Substitutes): Let $\boldsymbol{x, x' \in X}$. For any $\boldsymbol{\varepsilon > 0}$, there is a $\boldsymbol{c_0 \geq 0}$ such that for any $\boldsymbol{c, c' \geq c_0}$,

(*) $\boldsymbol{(x, c) \sim_i (x', c + \alpha) \;\&\; (x, c') \sim_i (x', c' + \alpha') \Longrightarrow |\alpha - \alpha'| < \varepsilon}$.

C4: There is an $x_0 \in X$ such that $(x_0, 0) \precsim_i (x, 0)$ for all $x \in X$.

Theorem (Approximate Quasi-Linearity). Let $\precsim_i$ be a preference relation on $X \times R_+$ satisfying C4. Then, $\precsim_i$ satisfies C0-C2 and C3

$\Leftrightarrow$ there are two functions $U_i: X \times R_+ \to R$ and $u_i: X \to R$ such that

- for all $(x, c), (x', c') \in X \times R_+$, $(x, c) \precsim_i (x', c') \Leftrightarrow U_i(x, c) \leq U_i(x', c')$,
- for all $(x, c), (x', c') \in X \times R_+$, if $U_i(x, c) \leq U_i(x', c')$, then $U_i(x, c + \alpha) = U_i(x', c')$ for some is an $\alpha \geq 0$,
- for any $x \in X$, there is some $c_0 \geq 0$ such that
$$|U_i(x, c) - (u_i(x) + c)| < \varepsilon \text{ for all } c \geq c_0.$$

- C3 is a Cauchy condition: a sequence $\{a_\nu\}$ converges to some $a \Leftrightarrow$ for any $\varepsilon > 0$, there is a ν_0 such that for any $\nu, \nu' \geq \nu_0$, $|a_\nu - a_{\nu'}| < \varepsilon$.

Cost-benefit analysis 2

- Cost-benefit analysis, e.g., to build a bridge (or not);
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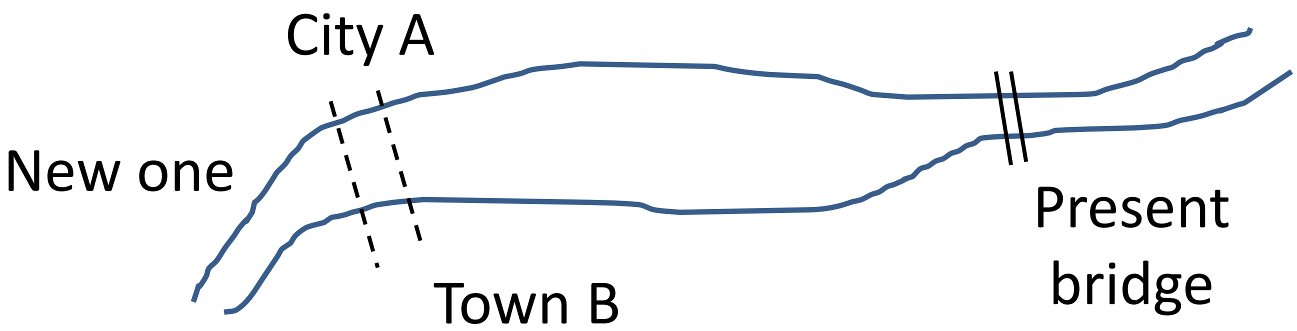


- How much can person in B pay for this bridge?
  - $U_i(0, I_i - u_i(0)) = U_i(b, I_i - u_i(b))$ for $i \in N$
  - $u_i(b) - u_i(0)$ depends upon I_i; but it has a limit as $I_i \to +\infty$ under C3.
  - These monetary "willingness to pays" can be summed up over the people.

Necessary Conditions from the theorem

Let x be fixed.

(1) $\lim_{c \to +\infty} \frac{U_i(x,c)}{c} = \lim_{c \to +\infty} \frac{u_i(x)+c}{c} = 1.$

Suppose $U_i(x,c) = u_i(x) + g_i(c)$. Then, (1) becomes

(1^*) $\lim_{c \to +\infty} \frac{U_i(x,c)}{c} = \lim_{c \to +\infty} \frac{u_i(x)+g_i(c)}{c} = \lim_{c \to +\infty} \frac{g_i{}'(c)}{1} = 1.$

Positive Example: $g_i(c) = c - \frac{1}{c+m}$ for $c \geq 0$, where m is a constant.

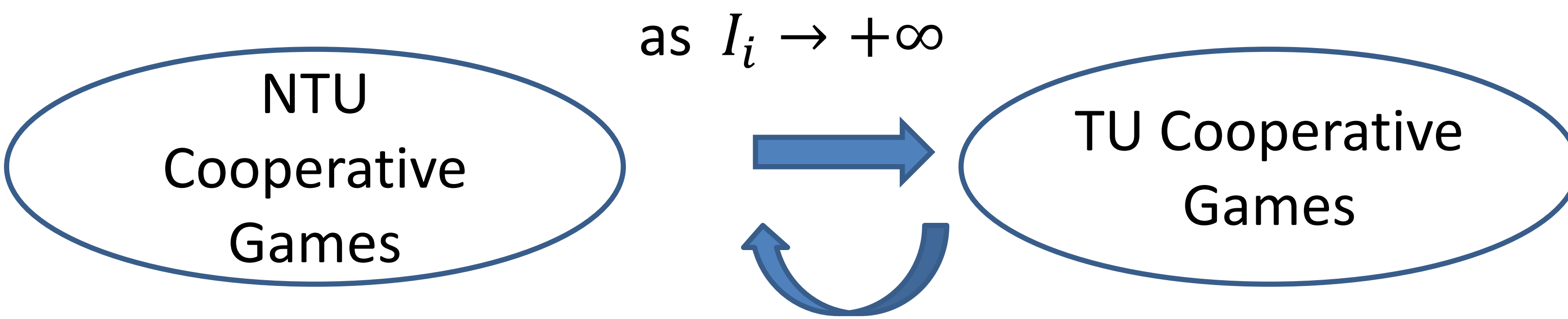
Negative Example: $g_i(c) = \sqrt{c}$ for $c \geq 0$. (1) does not hold.

- $U_i(x,c) = (u_i(x) + g_i(c))^2 = (u_i(x))^2 + 2u_i(x)\sqrt{c} + c$ satisfies (1), but doesn't satisfy C3.

TU Cooperative Game Theory

In which direction,

- from a large finite to the limit?



- from the limit to a large finite?
  - Resulting outcomes in the limit are approximated in the left.

Conclusions

- The condition for approximate quasi-linearity is presented.
- The QL assumption is not always obtained for large incomes.
- It is a condition for large incomes and may be regarded as a natural condition.
- Mathematically, however, many standard examples are eliminated.

- The paper gives another characterization in terms of the “normality” condition: as income increases, the quantity demanded increases, too.
  - Under this characterization, Lindahl-ratio equilibrium is considered.
- The paper gives a QL theory within expected utility, too.

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