Exchange Economies with Infinitely Many Commodities and a Saturated Measure Space of Consumers

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Abstract

The existence of competitive equilibria for exchange economies with a continuum of consumers over the commodity spaces ℓ^∞ and $ca(K)$ respectively will be proved. We define the economy as a measurable map from a *saturated* or *super-atomless* measure space to the space of consumers' characteristics following Aumann (1964), and prove the existence theorem without the convexity of preferences, applying a Fatou's lemma which has been recently obtained by the authors. Our model is considered to be a natural realization of the Aumann's thesis, or "many more agents than commodities". The representation and the realization of the distributional equilibria (Suzuki (2013a,b)) will be also discussed.

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1 Introduction

Several mathematical economists have tried to extend the results of Aumann (1964 and 66) for an economic model with an atomless measure space $(A, \mathcal{A}, \lambda)$ of consumers (a continuum of consumers) to the models with infinite dimensional commodity spaces. Among those, Bewley (1991), Noguchi (1997b) and Suzuki (2013c) proved the equilibrium existence theorems for the economies with the measure space of agents on the commodity space ℓ^∞. Khan-Yannelis (1991) and Noguchi (1997a) proved the same theorem for the continuum of consumers models on the commodity space which is a separable Banach space with an interior point of the positive orthant. In this paper, we will continue to explore this issue. In particular, we will prove the existence of competitive equilibria for exchange economies with the measure space of consumers and the commodity spaces ℓ^∞ and $ca(K)$, respectively.

In the previous papers (2013a and b) the second author proved the existence of competitive equilibria for an economy with a continuum of consumers with the commodity space ℓ^∞ (the model of infinite time horizon). He defined the economy as a probability measure μ on the set of agents' characteristics $\mathcal{P} \times \Omega$ (the coalitional form or the distribution form), where $\mathcal{P}$ is the set of consumers' preferences and Ω is that of initial endowments. Then the competitive equilibrium of this economy is also defined as a price vector $\boldsymbol{p} \in \ell^1$ and a probability measure ν on $X \times \mathcal{P} \times \Omega$, where the set $X \subset \ell^\infty$ is a consumption set which is assumed to be identical among all consumers. These definitions of the economy and the competitive equilibrium on it were first proposed by Hart and Kohlberg (1974), and applied to the model on the space $ca(K)$ by Mas-Colell (1975) and Jones (1983) (the model of the commodity differentiation)[1]. Although this is a natural formulation for an economy with the continuum of consumers and the infinitely many commodities, the economy and equilibria of it are defined only distributionally, hence all of the information on the individual level will be lost from them. If we want to obtain the individual information on them, we have to set up the model of the individual form of Aumann which defines the economy by a measurable map $\mathcal{E} : A \to \mathcal{P} \times \Omega$. The cost for obtaining the precise information, however, has seemed to be the convexity on the preferences of the consumers, which is not needed for economies of the coalitional form (see Mas-Colell (1975), Jones (1983) and Suzuki (2013a and b)).

The mathematical reason for the convexity assumption has been well known. Up to now, no direct proofs for an equilibrium existence theorem with the continuum of consumers and the infinite dimensional commodity space have been available. We have to resort to some approximations. In Suzuki (2013c) for instance, the proof of the existence of equilibrium $(\boldsymbol{p}, \xi)$ is carried out by approximating the large-infinite dimensional economy by large-finite dimensional sub-economies with equilibria $(\boldsymbol{p}_n, \xi_n)$. In the course of the approximation $\int_A \xi_n(a) d\lambda \to \int_A \xi(a) d\lambda$, we need the Fatou's lemma. On the finite dimensional spaces, we have the limit function of the sequence of allocations is obtained from the limit set of that sequence, namely that $\xi(a) \in Ls(\xi_n(a))$. The

[1] Bewley (1991) also used this approach.

infinite dimensional versions of Fatou's lemma, however, hold only "approximately", or they have only ensured that $\xi(a) \in \overline{co}Ls(\xi_n(a))$, where $\overline{co}S$ means the closed convex hull of a set S. Hence the convex valuedness of the demand correspondences themselves has been considered to be necessary.

Bewley indeed assumed the strong convexity of preferences, Khan-Yannelis and Suzuki also assumed that the preferences are convex (in the usual sense). Noguchi assumed that a commodity vector does not belong to the convex hull of its preferred set. Ostroy and Zame (1994) constructed a continuum exchange economy on the commodity space $ca(K)$ of the individual form. However, they also needed the convexity of preferences in order to prove the existence of equilibrium by exactly the same reason explained above. The convexity assumptions obviously weaken the impact of the Aumann's classical result which revealed the "convexfying effect" of large numbers of the economic agents.

Rustichini and Yannelis (1991) was probably the first paper which tackled the equilibrium existence problem for a model of the individual form on an infinite dimensional commodity space without the convexity of preferences. They concluded that in order to obtain any Fatou-type theorem (lemma), one has to have "many more agents than commodities". According to a footnote of Mertens (1991, p.189), this "many more agents than commodities" thesis seemingly at first addressed by Aumann himself in the context of the core-equivalence theorem[2], hence we call it "Aumann's thesis". In the Rustichini-Yannelis paper, the Aumann's thesis was stated as follows.

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space (of consumers) and $\mathcal{A}_E = \{A \cap E | \ A \in \mathcal{A}\}$ the sub-σ algebra of $\mathcal{A}$ restricted to $E \in \mathcal{A}$. We denote the restriction of λ to $\mathcal{A}_E$ by λ_E. Recall that for any (real) vector space, an algebraic Hamel basis exists. The cardinality of any Hamel basis of a vector space L is the same, and we denote it $dim(L)$. Rustichini and Yannelis proposed the next condition which is their version of the Aumann's thesis.

(RY) For any $E \in \mathcal{A}$ with $\lambda(E) > 0$, $dim(L^\infty_E(\lambda)) > dim(L)$,

where $L^\infty_E(\lambda))$ is the space of essentially bounded functions on E and L is the commodity space. Note that this condition involves the both spaces of consumers and the commodities.

Podczeck criticized this condition that "one may wish to interpret an atomless measure spaces as an idealization of a large but finite number of them. From this point of view, it is preferable to keep a measure space of agents 'small' (1997, p.386)." We admit the Podczeck's criticism to be reasonable. We would like to, however, point out that he had to pay the cost for requiring the extrinsic and an artificial mathematical structure of the measure space. He took a quotient space $\mathcal{T} = A/\sim$ where we set $a \sim a'$ if and only if $(\succsim_a, \omega(a)) = (\succsim_{a'}, \omega(a'))$. Then $\mathcal{T}$ represents the set of the equivalent classes of the consumers with the same "type". In other words, $\mathcal{T} = \{\tau = \mathcal{E}^{-1}(\succsim, \omega) | \ (\succsim, \omega) \in \mathcal{P} \times \Omega\}$. He postulates that for each $\tau \in \mathcal{T}$, the population measure λ is decomposed into a family of $\{\lambda_\tau\}$ and each measure λ_τ is concentrated on the set $\mathcal{E}^{-1}(\succsim, \omega)$ of consumers in A associated with the type $(\succsim, \omega)$. The essential part of Podczeck's version of the thesis is that every λ_τ is an atomless probability measure.

[2]Aumann was indeed correct. See Tourky and Yannelis (2001) and Podczeck (2005).

Some theorists including Podczeck himself in particular, however, have recognized that the Podczeck's condition contains a correct path to the true statement of the Aumann's thesis. This is accomplished by considering "Maharam types", instead of simply the types of consumers. The Maharam type of a measure space $(A, \mathcal{A}, \lambda)$ is the smallest cardinal of generating sub-algebras of $\mathcal{A}$ and it indicates in some sense the "size" of $\mathcal{A}$. For instance, if a (finite) measure space $(A, \mathcal{A}, \lambda)$ is atomless, its Maharam types of sub-measure spaces $(E, \mathcal{A}_E, \lambda_E)$ are infinite for any $E \subset A$ with $\lambda(E) > 0$ (Fact 1 in Section 2).

In the present paper, we would like to propose that the Aumann's thesis is manifestly represented when we set a *saturated* (or *super-atomless*[3]) measure space of consumers. A measure space $(A, \mathcal{A}, \lambda)$ is saturated if its Maharam types of sub-measure spaces $(E, \mathcal{A}_E, \lambda_E)$ are uncountable for any $E \subset A$ with $\lambda(E) > 0$ (Definition 1 in Section 2). Hence the saturated measure space strengthens the atomless measure space (hence called super-atomless). We will give some histrical remarks related with it in Section 7.

As Podczeck (2008) pointed out, the saturated measure space itself does not necessarily have an extraordinarily large cardinality. This can be explained by the non-trivial Loeb measure spaces (Keisler and Sun (2002)) which are important examples of the saturated measure spaces. It is possible for some non-trivial Loeb measure spaces to have cardinality of the continuum, hence can be identified with the unit interval on the real line. Keisler-Sun showed that for any atomless Loeb space $(A, \mathcal{A}, \lambda)$, Z any Polish space, $\mathcal{E} : A \to Z$ any $\mathcal{A}$-$\mathcal{B}(Z)$-measurable mapping, the set $\mathcal{E}^{-1}(z)$ has cardinality of greater than or equal to the continuum for almost all z (where "for almost all" means in the image measure on Z of $\mathcal{E}$). This captures essentially the Podczeck version of thesis when $Z = \mathcal{P} \times \Omega$. The Aumann's thesis is now embodied intrinsically in the measure space of consumers, rather than a condition imposed on it from outside; it is realized naturally in our models.

We believe that our choice of the commodity spaces, ℓ^∞ and $ca(K)$, will be justified in that these spaces have clear and definitive interpretations. Moreover, the model settings and the procedures to prove the existence theorems have been well established for these spaces. In particular, those authors who studied on the space ℓ^∞ or $ca(K)$, namely Bewley, Mas-Colell, Jones, Noguchi, Ostroy-Zame and Suzuki all worked with the resource condition in terms of the Gelfand integral. Therefore we need the Fatou's lemma for the Gelfand integrable maps. This has been indeed obtained by the recent paper by Khan-Sagara-Suzuki (2013), which proved an exact version of the Fatou's lemma for Gelfand integrable maps on a saturated measure space. The statement of their version of the lemma is "exact" in the sense of the finite dimensional version, or $Ls(\int_A \xi_n(a) d\lambda) \subset \int_A Ls(\xi_n(a)) d\lambda$, hence we can discard the convexity of preferences.

The saturated measure space also casts the light on the relation between the individual form and the distributional form of the economies and their equilibria. When the space of the characteristics $\mathcal{P} \times \Omega$ is a complete separable metric space, the distributional economy μ has a representation (Fact

[3]The name "super-atomless" was coined by Podczeck (2008).

2 in Section 2). This is the case when the consumption set X is a weak* compact subset of a dual Banach space, and in this situation, the distributional equilibrium also has a representation $(\xi, \mathcal{E})$. An important question is the following. Let an distributional economy μ and its equilibrium ν be given. For each representation $\mathcal{E}$ of μ, do we have an equilibrium allocation map ξ such that $(\xi, \mathcal{E})$ represents ν? Generally the answer is no. As stated in the second paragraph, all of the information of $\mathcal{E}$ are lost from μ. However, we will obtain the affirmative answer when the measure space is saturated (Proposition 1 in Section 4). Hence the distributional equilibrium ν does not loose any individual information when one works with the saturated economies; the individual and the distributional equilibria are equivalent in a strong sense for the saturated measure spaces. We will discuss more on the realization of the distributional equilibria in Section 4.

Those authors who studied on the space ℓ^∞, namely Bewley (1991), Noguchi (1997b) and Suzuki (2013a, b and c) all assumed that the consumption set is identical for all consumers and is a convex and bounded subset of the positive orthant. This is a regrettable situation compared to Khan-Yannelis (1991), Noguchi (1997a) and Rustichini-Yannelis (1991) in which the consumption sets can depend upon the individuals, and are assumed to be integrably bounded. Our Fatou's lemma can relax the uniform boundedness of consumption sets, namely our consumption sets are dependent on the consumers essentially in the same way as those of Khan-Yannelis, Noguchi and Rustichini-Yannelis. For the assumption on the initial endowments, however, we will depart from most of those authors. Indeed, Bewley (1991), Khan-Yannelis (1991), Noguchi (1997a, b), Rustichini-Yannelis (1991) and Suzuki (2013b, c) assumed that almost all consumers have their initial endowments in the (norm) interior of the consumption set. Obviously such an assumption is very strong in the economies with a continuum of traders.[4] We will adopt the irreducible condition initiated by McKenzie (1959)[5]. Notice that these remarks are not required for the models on $ca(K)$, except that all consumers are assumed to have the nonnegative orthant of $ca(K)$ as their common consumption set. See Section 5.

The Fatou's lemma will be presented in the next section (Fact 13) where all of the mathematical results necessary for the proofs are given and the notations are fixed. This result will play a key role in the present paper and it will provide a natural proof for the existence theorems. We say that the proofs are natural, meaning that we need no new or tricky ideas. All ideas for our lemmas have been well known. All we have to do is to exploit and combine them in an appropriate manner for each of our settings. The models will be presented through Sections 3, 4 and 5. Section 6 will be devoted to the proofs. Section 7 concludes.

[4] Precisely, Khan-Yannelis, Noguchi, Rustichini-Yannelis assumed that there exists $\zeta \in X$ such that $\omega - \zeta$ belongs to the (norm) interior of X.

[5] Suzuki (2013a) also applied this condition for the model with the continuum of consumers and the infinite time horizon.

2 Mathematical Preliminaries

2.1 Saturated Measure Spaces

A measure algebra is a pair $(\mathcal{A}, \lambda)$, where $\mathcal{A}$ is a Boolean σ-algebra with binary operations $\wedge$ and $\vee$, a unary operation ^{c} and λ is a real valued function satisfying the following conditions:(i) $\lambda(B) = 0$ if and only if $B = \emptyset$, where $\emptyset = A^c$ and $A = \emptyset^c$ are the smallest and the largest elements in $\mathcal{A}$, respectively; (ii) $\lambda(\wedge_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \lambda(E_n)$ for every sequence $\{E_n\}$ in $\mathcal{A}$ with $E_n \cap E_m = \emptyset$ whenever $m \neq n$. A map $\Phi : \mathcal{A} \to \mathcal{B}$ between measure algebras $(\mathcal{A}, \lambda)$ and $(\mathcal{B}, \mu)$ is called homomorphism if it is one to one, $\Phi(A^c) = \Phi(A)^c$, $\Phi(A \vee B) = \Phi(A) \vee \Phi(B)$ and $\lambda(A) = \mu(\Phi(A))$. Measure algebras $(\mathcal{A}, \lambda)$ and $(\mathcal{B}, \mu)$ are isomorphic if there exists a homomorphism which is onto.

A subalgebra of $\mathcal{A}$ is a subset of $\mathcal{A}$ which contains A and is closed under the Boolean operation $\wedge$, $\vee$ and ^{c}. The order $\leq$ on $\mathcal{A}$ is given by $B \leq C$ if and only if $B = B \wedge C$. A subalgebra $\mathcal{U}$ of $\mathcal{A}$ is order-closed with respect to $\leq$ if any nonempty upwards directed subsets of $\mathcal{U}$ with their supremum in $\mathcal{A}$ have the supremum in $\mathcal{U}$. A subset $\mathcal{U} \subset \mathcal{A}$ completely generates $\mathcal{A}$ if the smallest order closed subalgebra in $\mathcal{A}$ containing $\mathcal{U}$ is $\mathcal{A}$ itself. The Maharam type of $(\mathcal{A}, \lambda)$ is the smallest cardinal of any subset $\mathcal{U}$ which completely generates $\mathcal{A}$.

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space. We define an equivalence relation on A by $E \sim F$ if and only if $\lambda(E \Delta F) = 0$, where $E \Delta F = (E \wedge F^c) \vee (E^c \wedge F)$. The quotient space is denoted by $\hat{A} = A/\sim$. The equivalence class represented by $E \in \mathcal{A}$ is denoted $\hat{E}$. Then the lattice operation and the unary operation ^{c} is defined naturally on $\hat{A}$, $\hat{E} \vee \hat{F} = \widehat{E \cup F}$, $\hat{E}^c = \widehat{(E^c)}$. The pair $(\hat{A}, \hat{\lambda})$ is a measure algebra associated with $(A, \mathcal{A}, \lambda)$, where $\hat{\lambda}(\hat{E}) = \lambda(E)$. Moreover, $(\hat{A}, \hat{\lambda})$ becomes a complete metric space by the metric $\rho(E, F) = \lambda(E \Delta F)$ (see Aliprantis-Border (2006, Lemma 13.13)). The measure algebra $(\hat{A}, \hat{\lambda})$ is separable if it is a separable metric space. The Maharam type of $(A, \mathcal{A}, \lambda)$ is defined to be that of $(\hat{A}, \hat{\lambda})$.

Let $\mathcal{A}_E = \{A \cap E | \ A \in \mathcal{A}\}$ the sub-σ algebra of $\mathcal{A}$ restricted to $E \in \mathcal{A}$. We denote the restriction of λ to $\mathcal{A}_E$ by λ_E, or $\lambda_E(B) = \lambda(B)$ for every $B \in \mathcal{A}_E$. A finite measure space $(A, \mathcal{A}, \lambda)$ is (Maharam-type) homogeneous if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is equal to $(A, \mathcal{A}, \lambda)$. It is easy to see (e.g, Podczeck (2008 p.838))

Fact 1. A finite measure space $(A, \mathcal{A}, \lambda)$ is atomless if and only if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is infinite.

This fact motivates the next definition.

Definition 1. A finite measure space $(A, \mathcal{A}, \lambda)$ is *saturated* (or *super-atomless*) if for every $E \in \mathcal{A}$ with $\lambda(E) > 0$, the Maharam type of $(E, \mathcal{A}_E, \lambda_E)$ is uncountable.

Notice that the saturated measure spaces are not necessarily Maharam homogeneous (see also Section 7). Typical examples of the (homogeneous) saturated measure spaces are the atomless Loeb space (Loeb (1975)), the product spaces of the form $[0, 1]^{\mathbf{m}}$ and $\{0, 1\}^{\mathbf{m}}$, where $\mathbf{m}$ is an uncountable cardinal, $[0,1]$ equipped with the Lebesgue measure, and $\{0,1\}$ the "half-half" measure. Measure algebras of $[0,1]^{\mathbf{m}}$ and $\{0,1\}^{\mathbf{m}}$ are homogeneous with their Maharam type $\mathbf{m}$, and are isomorphic whenever $\mathbf{m}$ is infinite cardinal (see Fremlin (1975, Theorems 331I and 331K)), and they are separable if and only if $\mathbf{m}$ is countable.

For a complete separable metric (or Banach) space X, we denote the Borel σ-algebra of X which is defined as the σ-algebra generated by open subsets of X by $\mathcal{B}(X)$ and the set of Borel probability measures by $\mathcal{M}(X)$. Let f be a Borel measurable map from $(A, \mathcal{A}, \lambda)$ to X. The direct image measure $\lambda \circ f^{-1}$ is denoted by $f_*\lambda$. The operator $_*\lambda$ is a map from the set of all measurable maps of A to X which is denoted by $L^0(A, X)$ to $\mathcal{M}(X)$. We then have (Keisler-Sun (2009), Lemma 2.1)

Fact 2. The map $_*\lambda : L^0(A, X) \to \mathcal{M}(X)$ is surjective.

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, X and Y be complete separable metric spaces. The next definition reveals a crucial property of the saturated spaces.

Definition 2. A finite measure space $(A, \mathcal{A}, \lambda)$ is said to satisfy the *saturation property* for a measure $\mu \in \mathcal{M}(X \times Y)$ if for every measure $\mu \in \mathcal{M}(X \times Y)$ and measurable function f on A with $f_*\lambda = \mu_X$, there exists a measurable function g on Y which satisfies $(f, g)_*\lambda = \mu$,

where μ_X means the marginal distribution of μ on X. Let $L^1(\lambda)$ be the set of all λ-integrable functions on A,

$$L^1(\lambda) = \left\{ f : A \to \mathbb{R} \middle| \int_A |f(a)| d\lambda < +\infty \right\}.$$

Denote by $L^1_E(\lambda)$ be the vector subspace of $L^1(\lambda)$ whose element is a restriction of each function in $L^1(\lambda)$ to E. It is well known that $(\hat{A}, \hat{\lambda})$ is separable if and only if $L^1(\lambda)$ is separable (Aliprantis-Border (2006, Lemma 13.14)). The next characterization of the saturation is well known (see Fremlin (2012) and Keisler-Sun (2009)) and would be sometimes useful. Also the condition (c) could be interesting when compared to the Rustichini-Yannelis condition (RY) (see Section 1).

Fact 3. Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, X and Y be complete separable metric spaces. Then the following conditions are equivalent.

(a) $(A, \mathcal{A}, \lambda)$ is saturated,

(b) $(A, \mathcal{A}, \lambda)$ is atomless, and satisfies the saturated property for every $\mu \in \mathcal{M}(X \times Y)$,

(c) $L^1_E(\lambda)$ is non-separable for every $E \in \mathcal{A}$ with $\lambda(E) > 0$.

Let $I = [0,1]$ and $(I, \mathcal{I}, \bar{\ell})$ be the usual Lebesgue space. Keisler-Sun (2009) proved

Fact 4. Let $(A, \mathcal{A}, \lambda)$ be a finite measure space, X and Y be complete separable metric spaces and $f : A \to X$, $g : A \to Y$ be measurable maps, and assume that $f_*\lambda$ is atomless. Suppose that $(A, \mathcal{A}, \lambda)$ has the saturation property for $(f, g)_*\lambda$ but the Lebesgue space $(I, \mathcal{I}, \bar{\ell})$ does not. Then $(A, \mathcal{A}, \lambda)$ is saturated.

2.2 Some Banach Spaces

First we simply recall that the space of all bounded sequences

$$\ell^\infty = \left\{ \xi = (\xi^t) \middle|\ sup_{t \geq 1} |\xi^t| < +\infty \right\},$$

is a non-separable Banach space with respect to the norm $\|\xi\| = sup_{t\geq 1}|\xi^t|$ for $\xi \in \ell^\infty$ (Royden (1988)) with the dual space

$$ba = \left\{ \pi : 2^{\mathbb{N}} \to \mathbb{R} \middle| sup_{E \subset \mathbb{N}} |\pi(E)| < +\infty,\ \pi(E \cup F) = \pi(E) + \pi(F) \ \text{ whenever } \ E \cap F = \emptyset \right\}$$

which is the space of bounded and finitely additive set functions on $\mathbb{N}$. Let $\ell^\infty_+ = \{\xi \in \ell^\infty |\ \xi \geq \mathbf{0}\}$ be the non-negative orthant of ℓ^∞.

Similarly the dual space of the space of all summable sequences,

$$\ell^1 = \left\{ \boldsymbol{p} = (p^t) \middle|\ \sum_{t=1}^{\infty} |p^t| < +\infty \right\},$$

which is a separable Banach space with the norm $\|\boldsymbol{p}\| = \sum_{t=1}^{\infty} |p^t|$ is ℓ^∞. The non-negative orthant ℓ^1_+ is defined similarly as ℓ^∞_+. It is easy to see that the space ℓ^1 is isomorphic to the subspace ca of ba,

$$ca = \left\{ \pi \in ba \middle| \pi(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} \pi(E_n) \ \text{ whenever } \ E_i \cap E_j = \emptyset \ (i \neq j) \right\},$$

which is the space of the bounded and countably additive set functions on $\mathbb{N}$. For $\xi = (\xi^t) \in \mathbb{R}^\ell$ or ℓ^∞ or ℓ^1, $\xi \geq \mathbf{0}$ means that $\xi^t \geq 0$ for all t and $\xi > 0$ means that $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$. $\xi \gg \mathbf{0}$ means that $\xi^t > 0$ for all t. Finally for $\xi = (\xi^t) \in \ell^\infty$, we denote by $\xi \ggg \mathbf{0}$ if and only if there exists an $\epsilon > 0$ such that $\xi^t \geq \epsilon$ for all t.

The set function $\pi \in ba$ is called purely finitely additive if $\rho = 0$ whenever $\rho \in ca$ and $0 \leq \rho \leq \pi$. The relation between the ba and ca is made clear by the next fundamental theorem,

Fact 5 (Yosida-Hewitt (1952)). If $\pi \in ba$ and $\pi \geq 0$, then there exist set functions $\pi_c \geq 0$ and $\pi_p \geq 0$ in ba such that π_c is countably additive and π_p is purely finitely additive and satisfy $\pi = \pi_c + \pi_p$. This decomposition is unique.

Let (K, d) be a compact metric space. The space $ca(K)$ is the set of bounded countably additive set functions (signed measures) on K,

$$ca(K) = \left\{ \xi : \mathcal{B}(K) \to \mathbb{R} \middle|\ sup_{E \subset \mathbb{N}} |\xi(E)| < +\infty,\ \xi(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \xi(E_i) \right.$$
$$\left. \text{whenever } \ E_i \cap E_j = \emptyset \ (i \neq j) \right\}.$$

Then $ca(K)$ is a Banach space with respect to the norm

$$\|\xi\| = sup \left\{ \sum_{i=1}^{n} |\xi(E_i)| \middle|\ E_i \cap E_j = \emptyset \ \text{ for } \ i \neq j, n \in \mathbb{N} \right\} \ \text{ for } \ \xi \in ca(K).$$

Let $C(K)$ be the set of all continuous functions on K. $C(K)$ is also a separable Banach space with respect to the norm $\|\boldsymbol{q}\| = sup\{|\boldsymbol{q}(t)|| \ t \in K\}$ for $\boldsymbol{q} \in C(K)$. The Riesz representation theorem (Royden (1988) p.357) asserts that the dual space of $C(K)$ is $ca(K)$, or $C^*(K) = ca(K)$. For $\xi \in ca(K)$, $\xi \geq \mathbf{0}$ means that $\xi(B) \geq 0$ for every $B \in \mathcal{B}(K)$. $\xi > \mathbf{0}$ means that $\xi \geq \mathbf{0}$ and $\xi \neq \mathbf{0}$. The non-negative orthant of $ca(K)$ or $ca_+(K) = \{\xi \in ca(K)| \ \xi \geq \mathbf{0}\}$ is nothing but the set of Borel measures $\mathcal{M}(K)$ on K. For $t \in K$, the Dirac measure δ_t is defined by $\delta_t(E) = 1$ when $t \in E$, $\delta_t(E) = 0$ when $t \notin E$. Since (K, d) is a compact metric space, it is separable. Hence there exists a countable dense subset $\{t_1, t_2 \dots\}$ of K. Let $LS(t_1 \dots t_n)$ be the linear space spanned by $\{\delta_{t_1} \dots \delta_{t_n}\}$. It is well known that the set $\cup_{n=1}^{\infty} LS(t_1 \dots t_n)$ is dense in $ca(K)$ in the weak* topology.

Bounded subsets of ℓ^∞ and $ca(K)$ are $\sigma(\ell^\infty, \ell^1)$ and $\sigma(ca(K), C(K))$-weakly compact respectively, namely that the weak*-closure of the sets are weak*-compact by the Banach-Alaoglu's theorem.

Fact 6 (Rudin (1991, pp.68-70). If L is a Banach space, then the unit ball of L^*, $B = \{\pi \in L^* | \|\pi\| \leq 1\}$ is compact in the $\sigma(L^*, L)$-topology. Moreover, if L is a separable Banach space, then norm-bounded subset of L^* is a compact metric space.

Let $\{K^n\}$ be an increasing sequence of closed subsets of a compact metric space K converging to K in the topology of closed convergence. If $\boldsymbol{q}_n : K^n \to \mathbb{R}$ is continuous, we will write $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ if $\boldsymbol{q} \in C(K)$ and for every subsequence n_k and $t^{n_k} \in K^{n_k}$ with $t^{n_k} \to t$, $\boldsymbol{q}(t^{n_k}) \to \boldsymbol{q}(t)$. We have

Fact 7 (Mas-Colell (1975)). Let $\{\xi_n\}$ be a bounded sequence in $ca(K)$ with $support(\xi_n) \subset K^n$, and $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$. Then $\boldsymbol{q}_n \xi_n \to \boldsymbol{q}\xi$.

Let $(K^n, \boldsymbol{q}_n)$ be a sequence as above. We will say that it is equi-continuous if for all $\epsilon > 0$ there is a $\delta > 0$ such that for all $t, s \in K^n$ with $d(t, s) \leq \delta$, $|\boldsymbol{q}_n(t) - \boldsymbol{q}_n(s)| \leq \epsilon$. Mas-Colell (1975) proved

Fact 8. Let $\{K^n\}$ be a sequence of closed sets of a compact metric space K with $K^n \subset K^{n+1} \subset \dots \to K$ in the topology of closed convergence and $\{\boldsymbol{q}_n\}$ a sequence in $C(K)$ with $\|\boldsymbol{q}_n\| \leq \rho$ for all n and for some fixed $\rho > 0$. If $(K^n, \boldsymbol{q}_n)$ is equi-continuous, then there is a subsequence n_k and $\boldsymbol{q} \in C(K)$ with $(K^{n_k}, \boldsymbol{q}_{n_k}) \to (K, \boldsymbol{q})$.

2.3 Integrals of Vector Valued Maps

Let $(A, \mathcal{A}, \lambda)$ be a finite measure space. A map $f : A \to \ell^\infty$ is said to be weak* measurable if for each $\boldsymbol{p} \in \ell^1$, $\boldsymbol{p}f(a)$ is measurable. A map $f : A \to \ell^\infty$ is weakly measurable if for each $\pi \in ba$, $\pi f(a)$ is measurable. A weak* (weakly) measurable map $f(a)$ is said to be Gelfand (Pettis) integrable if there exists an element $\xi \in \ell^\infty$ such that for each $\boldsymbol{p} \in \ell^1 (\pi \in ba)$, $\boldsymbol{p}\xi = \int \boldsymbol{p}f(a)d\lambda$ $(\pi\xi = \int \pi f(a)d\lambda)$. The vector ξ is denoted by $\int f(a)d\lambda$ and called Gelfand (Pettis) integral of f. Obviously if $f : A \to \ell^\infty$ is Pettis integrable, then it is also Gelfand integrable, and the both integrals coincide.

Recall that a topological space is a Suslin space if and only if it is the image of a continuous map from a complete and separable metric space (Polish space). A separable Banach space L is a Suslin space, and its dual space L^* endowed with the weak topology is also a Suslin space. It is true that for a Suslin space L, the space which is equipped with a weaker Hausdorff topology has the same Borel sets as L (Thomas (1975, p.67)). Hence ℓ^∞ with the weak topology $\sigma(\ell^\infty, ba)$ is a Suslin space, and so is it with the weak* topology $\sigma(\ell^\infty, \ell^1)$ which is weaker than the weak topology. A map from a measure space $(A, \mathcal{A}, \lambda)$ to a locally convex Suslin space is Borel measurable if and only if it is weakly measurable (Thomas (1975, Theorem 1)). Hence a Borel measurable map $f : (A, \mathcal{A}, \lambda) \to (\ell^\infty, \mathcal{B}^*(\ell^\infty))$ is weakly measurable, where $\mathcal{B}^*(\ell^\infty)$ is the set of Borel sets with respect to the weak* topology.

Similarly, a map $f : A \to ca(K)$ is said to be weak*-measurable if for each $\boldsymbol{q} \in C(K)$, $\boldsymbol{q}f(a)$ is a measurable function on $(A, \mathcal{A}, \lambda)$. A weak*-measurable function $f(a)$ is said to be Gelfand integrable if there exists an element $\int_A f(a)d\lambda \in ca(K)$ such that for each $\boldsymbol{q} \in C(K)$, $\boldsymbol{q}\int_A f(a)d\lambda = \int_A \boldsymbol{q}f(a)d\lambda$. In particular, for every Borel set $B \in \mathcal{B}(K)$, the value of the measure $\int_A f(a)d\lambda$ at B is defined by $\int_A f(a)d\lambda(B) \equiv \int_A f(a)(B)d\lambda$. Indeed, let $\boldsymbol{q} \in C(K)$ and $\{B_i\}$ be a family of pairwise disjoint Borel sets in K. Then we have

$$\begin{aligned} \boldsymbol{q}\int_A f(a)(\cup_{i=1}^\infty B_i)d\lambda &= \int_A \boldsymbol{q}f(a)(\cup_{i=1}^\infty B_i)d\lambda \\ &= \int_A \boldsymbol{q}\sum_{i=1}^\infty f(a)(B_i)d\lambda = \boldsymbol{q}\sum_{i=1}^\infty \int_A f(a)(B_i)d\lambda, \end{aligned}$$

hence $\int_A f(a)d\lambda(\cup_{i=1}^\infty B_i) = \int_A f(a)(\cup_{i=1}^\infty B_i)d\lambda = \sum_{i=1}^\infty \int_A f(a)(B_i)d\lambda = \sum_{i=1}^\infty \int_A f(a)d\lambda(B_i)$.

Now the definition of the weak* (weak) measurability and the Gelfand (Pettis) integrals for general cases should be clear.

Fact 9 (Diestel and Uhl (1977, pp.53-4)). Let L be a Banach space and L^* its norm dual space. If $f : A \to L^*$ is weak*-measurable and $\boldsymbol{p}f(a)$ is integrable function for all $\boldsymbol{p} \in L$, then f is Gelfand integrable.

Fact 10 (Diestel and Uhl (1977, p.53)). Let L be a Banach space and L^* its norm dual space. If $f : A \to L$ is weakly measurable and $\pi f(a)$ is integrable function for all $\pi \in L^*$, then f is Pettis integrable.

Fact 11. Let $\{f_n\}$ be a sequence of Gelfand integrable functions from A to L^* which converges a.e to f in the weak* topology. Then it follows that $\int_A f_n(a)d\lambda \to \int_A f(a)d\lambda$ in the weak* topology.

Let X be a topological space and F_n a sequence of subsets of X. The topological limes superior $Ls(F_n)$ is defined by

$\xi \in Ls(F_n)$ if and only if there exists a sub-sequence F_{n_k} with $\xi_{n_k} \in F_{n_k}$ for all k and $\xi_{n_k} \to \xi \ (k \to \infty)$.

Consider a sequence of maps $\phi_n : A \to X$. For each $a \in A$, if $Ls(\phi_n(a)) \neq \emptyset$, we can define a map $Ls(\phi_n(\cdot)) : A \to X, a \mapsto Ls(\phi_n(a))$. The next theorem is due to Khan-Sagara-Suzuki (2013) which is an infinite dimensional version of the Fatou's lemma and plays a crucial role in our proof.

Fact 12. Let $(A, \mathcal{A}, \lambda)$ be a complete and finite measure space which is saturated, and L^* be the dual space of a separable Banach space L. Let $f_n : A \to L^*$ be a sequence of Gelfand integrable mappings from A to L^* such that there exists an integrable function $g(a)$ with $\sup_n \|f_n(a)\| \leq g(a)$ a.e.

Then there exists a Gelfand integrable map $f : A \to L^*$ with

$$\int_A f(a)d\lambda \in Ls\left(\int_A f_n(a)d\lambda\right) \text{ and } f(a) \in Ls(f_n(a)) \ a.e.$$

3 A Market with Infinite Time Horizon

The commodity vectors are represented by sequences $\xi = (\xi^t)$ in ℓ^∞. Here the economic meaning of each coordinate is of course that ξ^t is the amount of the commodity available at t. The price vector is assumed to be a vector in ℓ^1, hence the value of a commodity $\xi = (\xi^t) \in \ell^\infty$ evaluated by a price vector $\boldsymbol{p} = (p^t) \in \ell^1$ is given by the natural "inner product" $\boldsymbol{p}\xi = \sum_{t=1}^{\infty} p^t \xi^t$.

Let $(A, \mathcal{A}, \lambda)$ be a complete probability space of the consumers. Let $\beta(a)$ be an integrable function with $\beta(a) \geq \tilde{\beta}$ a.e for some positive number $\tilde{\beta}$. We will assume that the consumption set $X(a)$ of the consumer a is the set of nonnegative vectors whose coordinates are bounded by $\beta(a)$,

$$X(a) = \{\xi = (\xi^t) \in \ell^\infty | \ \mathbf{0} \leq \xi \leq \beta(a)\mathbf{1}\},$$

where $\mathbf{1} = (1, 1 \ldots)$. Of course the $\tilde{\beta} > 0$ is intended to be a very large number. Obviously $X(a)$ has a measurable graph, $\{(a, \xi) \in A \times \ell^\infty_+ | \ \xi \in X(a)\} \in \mathcal{A} \times \mathcal{B}^*(\ell^\infty_+)$, where $\mathcal{B}^*(\ell^\infty_+)$ be the set of Borel subsets of ℓ^∞_+ with respect to the weak* topology.

As usual, a preference $\succsim_a$ is a complete, transitive and reflexive binary relation on $X(a)$. We denote $(\xi, \zeta) \in \succsim_a$ by $\xi \succsim_a \zeta$. $\xi \prec_a \zeta$ means that $(\xi, \zeta) \notin \succsim_a$. Let $\mathcal{P}$ be the set of allowed consumption set-preference pairs,

$$\mathcal{P} = \{(X, \succsim) | \ X = \{\xi = (\xi^t) \in \ell^\infty | \ \mathbf{0} \leq \xi \leq \beta\mathbf{1}\} \text{ for some } \beta \geq \tilde{\beta}, \ \succsim \subset X \times X\}.$$

An endowment vector is an element of ℓ^∞. We denote the set of all endowment vectors by Ω and assume that it is of the form

$$\Omega = \{\omega = (\omega^t) \in \ell^\infty | \ \mathbf{0} \leq \omega^t \leq \tilde{\gamma}\mathbf{1}\},$$

for some $\tilde{\gamma} > 0$. We assume that $\tilde{\gamma} < \tilde{\beta}$. The set Ω is a compact metric space by Fact 6. Let $\tilde{X}$ be the nonnegative orthant of the commodity space, or $\tilde{X} = \ell^\infty_+$.

Assumption (MR). For each $a \in A$, we have a map which assigns a his/her preference $(X(a), \succsim_a) \in \mathcal{P}$ and is measurable in the sense that

$$\{(a, \xi, \zeta) \in A \times X(a) \times X(a) |\ \xi \succsim_a \zeta\} \in \mathcal{A} \times \mathcal{B}^*(\tilde{X}) \times \mathcal{B}^*(\tilde{X}).$$

An endowment assignment map ω is a Borel measurable map from A to $(\Omega, \mathcal{B}^*(\Omega))$, $a \mapsto \omega(a) \in \Omega$, where $\mathcal{B}^*(\Omega)$ is the Borel σ-algebra with respect to the weak* topology. Then it is weakly measurable (see Section 2.3), or for every $\pi \in ba$, $\pi\omega(a) : A \to \mathbb{R}$ is a measurable function. Note that by Fact 10 combined with the assumption for Ω, the map ω is Pettis integrable, hence it is Gelfand integrable.

The definition of the economy follows Aumann (1964 and 1966) and Hildenbrand (1974).

Definition 3. An economy $\mathcal{E}$ is a mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega$ defined by $a \mapsto ((X(a), \succsim_a), \omega(a))$. The economy is called saturated (or super-atomless) if the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

The assumptions for the preferences are:

Assumption (PR).

(*i*) For every $(X, \succsim) \in \mathcal{P}$, $\succsim \subset X \times X$ is complete, transitive and reflexive which is closed in $X \times X$ in the weak* topology,

(*ii*) (Monotonicity). For every $(X, \succsim) \in \mathcal{P}$, $\xi \prec \zeta$ whenever $\xi, \zeta \in X$ and $\xi < \zeta$.

As stated in Section 1, we do not need the convexity for the preferences. Compare with Bewley (1991), Noguchi (1997b), Suzuki (2013c) and others. The next assumption on the total endowment which says that it is in the norm interior of the consumption set is standard.

Assumption (TE) (Interior total endowment). $\int_A \omega(a) d\lambda \ggg \mathbf{0}$.

An allocation is a Gelfand integrable map $\xi : A \to \ell^\infty_+$ with $\xi(a) \in X(a)$ a.e. An allocation is said to be feasible if $\int_A \xi(a) d\nu \leq \int_A \omega(a) d\nu$. It is called exactly feasible if $\int_A \xi(a) d\nu = \int_A \omega(a) d\nu$. An economy $\mathcal{E}$ is called *irreducible* if it satisfies the next condition.

Assumption (IR) (Irreducibility). Let ξ be a feasible allocation, and $\{A_1, A_2\}$ a measurable partition of A with $\nu(A_i) > 0$, $i = 1, 2$ such that $\int_{A_2} \{\zeta \in X(a) |\ \xi(a) \prec_a \zeta\} d\nu \neq \emptyset$. Then there exists an allocation ϕ and a measurable set $A_3 \subset A_2$ with positive measure such that $\int_{A_1} (\omega(a) - \phi(a)) d\nu + \int_{A_3} \xi(a) d\nu \in \int_{A_3} \{\zeta \in X(a) |\ \xi(a) \prec_a \zeta\} d\nu$.

Remark. Note that the condition (IR) is concerned with the economy $\mathcal{E}$ itself, namely with the maps $(X(a), \succsim_a)$ and $\omega(a)$ both together. It was introduced by McKenzie (1959), and applied in a continuum of consumers model by Yamazaki (1981). It was also used for models with infinite time horizon by Boyd and McKenzie (1993) and Suzuki (2013a). See also McKenzie (2002). By Assumption (TE), one obtains that $\boldsymbol{p} \int_A \omega(a) d\lambda > 0$, where $\boldsymbol{p}$ is an equilibrium price vector. Hence $\lambda(\{a \in A|\ \boldsymbol{p}\omega(a) > 0\}) > 0$. The role of the irreducibility condition (IR) is that it makes $\lambda(\{a \in A|\ \boldsymbol{p}\omega(a) > 0\}) = 1$, in other words, "if some one has positive income, then every one does".

The next simple example could give some illustration for the economic environment consisting of the assumptions (PR), (TE) and (IR).

Example (Suzuki (2013a)). Suppose that $\beta(a) = \tilde{\beta} = 2$ a.e and $\tilde{\gamma} = 1$. Let $\{A_o, A_e\}$ be a measurable partition of A with $\nu(A_o) = \nu(A_e) = 1/2$. Consider the endowment assignment map defined by

$$\omega(a) = \begin{cases} (1, 0, 1, 0 \dots) & \text{for } a \in A_o, \\ (0, 1, 0, 1 \dots) & \text{for } a \in A_e. \end{cases}$$

The consumers in A_o has one unit of the consumption good in odd periods $t = 1, 3 \dots$ and the consumers in A_e has it in even periods. Obviously any consumer does not have its endowment vector in the interior of X. Since $\int_A \omega(a) d\nu = \int_{A_o} \omega(a) d\nu + \int_{A_e} \omega(a) d\nu = (1/2, 0, 1/2 \dots) + (0, 1/2, 0 \dots) = (1/2, 1/2 \dots)$, the assumption (TE) holds.

In order to check the condition (IR), first observe that the condition (IR) will be met under the monotonicity (PR)(ii) whenever $\int_{A_1} \omega(a) d\nu \gg \mathbf{0}$. So the only case we have to care is that $A_1 = A_o$. (Hence $A_2 = A_e$. The case $A_1 = A_e$ is similar.) Let $\xi(a)$ be a feasible allocation and define $A_3 = \{a \in A_2|\ \xi^t(a) < 2 \text{ for some odd } t\}$. Then $\nu(A_3) > 0$. For if not, one has $\int_{A_2} \xi^t(a) d\nu = 1$ for every odd t, hence $\int_A \xi^t(a) d\nu \geq \int_{A_2} \xi^t(a) = 1 > 1/2 = \int_A \omega^t(a) d\nu$, violating the feasibility. Under the monotonicity (PR)(ii), it is easy to obtain a map $\phi : A_1 \to X$ satisfying $\int_{A_1} (\omega(a) - \phi(a)) d\nu + \int_{A_3} \xi(a) d\nu \in \int_{A_3} \{\zeta \in X(a)|\ \xi(a) \prec_a \zeta\} d\nu$.

Definition 4. A pair $(\boldsymbol{p}, \xi)$ of a price vector $\boldsymbol{p} \in \ell^1_+$ with $\boldsymbol{p} \neq \mathbf{0}$ and an allocation ξ is called a competitive equilibrium of the economy $\mathcal{E}$ if the following conditions hold,

(E-1) $\boldsymbol{p}\xi(a) \leq \boldsymbol{p}\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\boldsymbol{p}\zeta \leq \boldsymbol{p}\omega(a)$ a.e,

(E-2) $\int_A \xi(a) d\lambda = \int_A \omega(a) d\lambda$.

The condition (E-1) says that almost all consumers maximize their utilities under their budget constraints. The condition (E-2) says that the total equilibrium allocation is exactly feasible.

Theorem 1. Let $\mathcal{E}$ be an economy which is saturated and satisfies the assumptions (MR), (PR), (TE) and (IR). Then there exists a competitive equilibrium $(\boldsymbol{p}, \xi)$ for $\mathcal{E}$.

4 Realization of Distributional Equilibria

In this section, we consider the model on the commodity space ℓ^∞ with $\beta(a) = constant$ a.e. Then $X(a) = X$ a.e is a compact metric space in the weak* topology. Note that $\mathcal{P}$ is also a compact metric space in the closed convergence topology (Hildenbrand (1974, p.19), hence $X \times \mathcal{P}\times\Omega$ is a compact metric space. The distributional economy (coalitional form, Suzuki (2013a,b)) is defined as follows.

Definition 5. A distributional economy is a probability measure μ on the measurable space $(\mathcal{P}\times\Omega, \mathcal{B}(\mathcal{P}\times\Omega))$. An atomless distributional economy μ is an atomless probability measure on $(\mathcal{P}\times\Omega, \mathcal{B}(\mathcal{P}\times\Omega))$.

Recall that the marginals of μ will be denoted by subscripts, for instance, $\mu_{\mathcal{P}}$ denotes the marginal on $\mathcal{P}$, and so on. A probability measure ν on $X \times \mathcal{P}\times\Omega$ is called an allocation distribution if $\nu_{\mathcal{P}\times\Omega} = \mu$. An allocation distribution is called (exactly) feasible if $\int_X i(\xi)d\nu_X = \int_\Omega i(\omega)d\mu_\Omega$. Since $i(\xi) = \xi$ for all ξ, hereafter we will denote $\int_X i(\xi)d\nu_X = \int_X \xi d\nu_X$, and so on. The Gelfand integrals $\int_X \xi d\nu_X$ and $\int_\Omega \omega d\mu_\Omega$ exist by Fact 9. The distributional equilibrium is defined as follows (Suzuki (ibid)).

Definition 6. A pair $(\boldsymbol{p}, \nu)$ of a price vector $\boldsymbol{p} \in \ell^1$ with $\boldsymbol{p} > \boldsymbol{0}$ and an allocation distribution ν on $X \times \mathcal{P}\times\Omega$ is called a competitive equilibrium of the economy μ if the following conditions hold,

(D-1) $\nu(\{(\xi, \succsim, \omega) \in X \times \mathcal{P}\times\Omega|\ \boldsymbol{p}\xi \le \boldsymbol{p}\omega \text{ and } \xi \succsim \zeta \text{ whenever } \boldsymbol{p}\zeta \le \boldsymbol{p}\omega\}) = 1$,

(D-2) $\int_X \xi d\nu_X = \int_\Omega \omega d\mu_\Omega$,

(D-3) $\nu_{\mathcal{P}\times\Omega} = \mu$.

The existence and the core equivalence of the distributional equilibria have been established by Suzuki (2013a). Let $(A, \mathcal{A}, \lambda)$ be an atomless probability measure space. For a measurable map $f : A \to \mathcal{P}\times\Omega$, recall that the direct image measure $\lambda\circ F^{-1}$ is denoted by $f_*\lambda$.

Definition 7. For an economy μ, a measurable map $\mathcal{E} : A \to \mathcal{P}\times\Omega$ such that $\mu = \mathcal{E}_*\lambda$ is called a representation of μ. The representation is called saturated if the measure space $(A, \mathcal{A}, \lambda)$ is saturated.

Note that a representation is not unique even if it exists. Since $\mathcal{P} \times \Omega$ is a compact metric space, the representations of μ exists by Fact 2. Moreover, since the saturated measure spaces are atomless, the saturated representations also exist. Similarly, for every allocation distribution ν, a measurable map $(\xi, \mathcal{E}) : A \to X \times \mathcal{P}\times\Omega$ which satisfies $\nu = (\xi, \mathcal{E})_*\lambda$ is the representation of ν. The map $\xi : A \to X$ is nothing but an allocation. The representations for ν also exist by the same reason for μ. We may call the existence of the representations for ν the *weak equivalence* of the individual and distributional equilibria (Khan-Rath-Haomiao-Zhang (2013)).

A fundamental problem is the realization of the distributional equilibrium or the *strong equivalence* of the two equilibria; given an equilibrium ν of an economy μ and an individual economy $\mathcal{E}$ which represents μ, can we obtain an allocation ξ such that $(\xi, \mathcal{E})$ represents ν? The answer is generally negative for atomless measure spaces of the consumers. For the saturated measure spaces, however, the positive answer will be the rule.

Theorem 2. Let $(A, \mathcal{A}, \lambda)$ be a saturated probability space. Let distributional economy μ and its equilibrium ν be given. For every individual economy $\mathcal{E} : A \to \mathcal{P} \times \Omega$ which represents μ, there exists an equilibrium allocation $\xi : A \to X$ such that $\nu = (\xi, \mathcal{E})_* \lambda$.

The concept of the symmetric equilibria in the next definition is due to Khan and Sun (1991).

Definition 8. The equilibrium ν is called symmetric if there exists a measurable map $\sigma : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\sigma)) = 1$, where $Graph(\sigma) = \{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega |\ \xi = \sigma(\succsim, \omega)\}$.

If an equilibrium is symmetric, then the consumers with the identical characteristics consume the identical consumption vector.

Definition 9. Let a distributional economy μ and its equilibrium ν be given. A probability space $(A, \mathcal{A}, \lambda)$ realizes ν, or $(A, \mathcal{A}, \lambda)$ is a realization of ν, if every individual economy $\mathcal{E} : A \to \mathcal{P} \times \Omega$ which represents μ has a measurable map $\xi : A \to X$ such that $\nu = (\xi, \mathcal{E})_* \lambda$.

The next result is already known for the large atomless games (Khan-Rath-Haomiao-Zhang (2013)).

Theorem 3. Let an atomless distributional economy μ and its equilibrium ν be given. Then the following conditions are equivalent.

(a) ν is symmetric,

(b) every atomless probability space is a realization of ν,

(c) every atomless non-saturated probability space is a realization of ν,

(d) the measure space $([0, 1], \mathcal{B}([0, 1]), \bar{\ell})$ (here $\bar{\ell}$ is the Lebesgue measure on the Borel σ-algebra) is a realization of ν.

Corollary. An atomless probability space $(A, \mathcal{A}, \lambda)$ realizes a non-symmetric equilibrium of an atomless distributional economy μ if and only if it is saturated.

5 A Market with Differentiated Commodities

Following Mas-Colell (1975), Jones (1983) and Ostroy-Zame (1994), the economic interpretation of K is that it is a space of the commodity characteristics. Hence each $t \in K$ represents the complete list of characteristics which describes the commodity. A (differentiated) commodity bundle ξ is defined as a signed measure on K, hence an element of $ca(K)$. In particular, the Dirac measure δ_t is the (one unit of) commodity bundle which contains a characteristics $t \in K$. In this section, all consumers are assumed to have an identical consumption set

$$X = \mathcal{M}(K) = \tilde{X}.$$

An initial endowment is assumed to be a nonnegative vector ω of $X = \mathcal{M}(K)$.

Let $(A, \mathcal{A}, \lambda)$ be a complete and saturated measure space of consumers. An endowment assignment is a Gelfand integrable map $\omega : A \to X$, $a \mapsto \omega(a)$. The assumption on the endowments for this economy is

Assumption (AE) (Adequate endowments). *support* $\left(\int_A \omega(a) d\lambda\right) = K$.

The assumption (AE) simply says that every commodity characteristics are available in the market.

As usual, a preference relation $\succsim \subset X \times X$ is a complete, transitive and reflexive binary relation on X, and we denote $(\xi, \zeta) \in \succsim$ by $\xi \succsim \zeta$. $\xi \prec \zeta$ means that $(\xi, \zeta) \notin \succsim$. Let $\mathcal{P}$ be a collection of allowed preference relations. We assume that $\mathcal{P}$ satisfies

Assumption (RS). For every $\rho > 1$, there exists $\eta > 0$ such that for all $\epsilon > 0$, for all $\xi \in X$, for all $\succsim \in \mathcal{P}$ and for all $s, t \in K$ with $d(s,t) < \eta$, $\xi + \epsilon\delta_s \prec \xi + \epsilon\rho\delta_t$.

Since $\mathcal{P} \subset \mathcal{F}(X \times X)$ by (i), we can endow $\mathcal{P}$ with the topology of closed convergence on $\mathcal{F}(X \times X)$[6].

Remark. The assumption (RS) says that each consumer is willing to accept any trade in which 'terms' (namely ρ) are strictly greater than one whenever the characteristics of the traded commodities are sufficiently close. Roughly speaking, (RS) requires that the marginal rate of substitutions between nearby commodities are uniformly (in ξ and $\succsim$) close to one. Jones (1983 and 1984) discussed that this assumption will hold if the utility functions representing the preferences are smooth and the derivatives satisfy some bounded conditions. See Jones (1983 and 1984) for more explanations for these conditions. If K is a finite set, then the condition (RS) is automatically satisfied, hence our results will cover the classical theorems of Aumann (1964 and 1966). Note that the parameter $\eta > 0$ is assumed to be taken uniformly over the preferences. This implicitly requires that the range of preferences under consideration satisfy some compactness condition, for instance, the set $\{\succsim(a) | \ a \in A\}$ is contained in some compact subset of $\mathcal{P}$, although we did not assume it explicitly.[7] In the term of Ostroy and Zame (1994), the market on $ca(K)$ which satisfies the assumption (RS) is "thick". For the discussions on "thick and thin" markets, see their paper.

As before, an economy $\mathcal{E}$ is a mapping $\mathcal{E} : A \to \mathcal{P} \times \Omega$ defined by $a \mapsto ((X, \succsim_a), \omega(a))$. Let $C_+(K) = \{\boldsymbol{q} \in C(K) | \ \boldsymbol{q} \geq \boldsymbol{0}\}$. A price vector is an element of $C_+(K)$. Then for $\xi \in X$ and

[6] Note that since X is not necessarily metrizable, we can not conclude that $\mathcal{F}(X \times X)$ is compact or metrizable.

[7] On the other hand, Mas-Colell (1975) and Jones (1983) assumed that the space $\mathcal{P}$ is compact.

$\boldsymbol{q} \in C_+(K)$, we denote $\boldsymbol{q}\xi = \int_K \boldsymbol{q}(t)d\xi(t)$. As usual, a measurable map $\xi : A \to X$ is called an allocation. The definition of the competitive equilibrium should be obvious.

Definition 10. A pair $(\boldsymbol{q}, \xi)$ of a price vector $\boldsymbol{q} \in C_+(K)$ with $\boldsymbol{q} \neq \mathbf{0}$ and an allocation ξ is called a competitive equilibrium of the economy $\mathcal{E}$ if the following conditions hold,

(E-1) $\boldsymbol{q}\xi(a) \leq \boldsymbol{q}\omega(a)$ and $\xi(a) \succsim \zeta$ whenever $\boldsymbol{q}\zeta \leq \boldsymbol{q}\omega(a)$ a.e,

(E-2) $\int_A \xi(a)d\lambda = \int_A \omega(a)d\lambda$.

The existence of equilibria for the model with the differentiated commodities is established by

Theorem 4. Let $\mathcal{E}$ be an economy which is saturated and satisfies the assumptions (MR), (PR), (RS) and (AE). Then there exists a competitive equilibrium $(\boldsymbol{q}, \xi)$ for $\mathcal{E}$.

6 Proofs

6.1 Proof of Theorem 1.

Let $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be the economy. For each $n \in \mathbb{N}$, let L^n be the canonical projection of ℓ^∞ to $\mathbb{R}^n$, $L^n = \{\xi = (\xi^t) \in \ell^\infty | \xi = (\xi^1, \xi^2 \dots \xi^n, 0, 0 \dots)\}$. We then define

$$X^n(a) = X(a) \cap L^n, \ \succsim_a^n = \succsim_a \cap (X^n(a) \times X^n(a)), \ \mathcal{P}^n = \mathcal{P} \cap 2^{L^n \times L^n}, \text{ and } \Omega^n = \Omega \cap L^n,$$

and for every $\omega = (\omega^1, \omega^2 \dots \omega^n, \omega^{n+1} \dots) \in \Omega$, we denote $\omega_n = (\omega^1, \omega^2 \dots \omega^n, 0, 0 \dots) \in \Omega^n$, the canonical projection of ω. Obviously $\omega_n \to \omega$ in the weak* topology. They induce finite dimensional economies $\mathcal{E}^n : A \to \mathcal{P}^n \times \Omega^n$ defined by $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$, $n = 1, 2 \dots$. Then we have

Lemma 1. . For each n, there exists a quasi-competitive equilibrium for the economy $\mathcal{E}^n$, or a price-allocation pair $(\pi_n, \xi_n(a))$ which satisfies

(Q-1n) $\pi_n \xi_n(a) \leq \pi_n \omega(a)$ and $\xi_n(a) \succsim_a \zeta$ whenever $\pi_n \zeta \leq \pi_n \omega_n(a)$ and $\pi_n \omega_n(a) > 0$ a.e,

(Q-2n) $\int_A \xi_n(a)d\lambda \leq \int_A \omega_n(a)d\lambda$.

Proof. See Khan and Yamazaki (1981), Proposition 2. ∎

Since $\omega_n(a) \to \omega(a)$ a.e, we have $\int_A \omega_n(a)d\lambda \to \int_A \omega(a)d\lambda$ by Fact 11. Without loss of generality, we can assume that $\pi_n \mathbf{1} = \sum_{t=1}^n p_n^t = 1$ for all n, where $\pi_n = (p_n^t)$ and $\mathbf{1} = (1, 1 \dots)$. Here we have identified $\pi_n \in \mathbb{R}_+^n$ with a vector in ℓ_+^1 which is also denoted by π_n as $\pi_n = (\pi_n, 0, 0 \dots)$. Since the set $\Delta = \{\pi \in ba_+ | \ \|\pi\| = \pi\mathbf{1} = 1\}$ is weak*-compact by the Alaoglu's theorem (Fact 6), we have a price vector $\pi \in ba_+$ with $\pi\mathbf{1} = 1$ and a subnet $(\pi_{n(\alpha)})$ such that $\pi_{n(\alpha)} \to \pi$ in the $\sigma(ba, \ell^\infty)$-topology. Since $\int_A \omega_{n(\alpha)}(a)d\lambda \to \int_A \omega(a)d\lambda$ and Ω is a compact metric space, we can extract from $\{\omega_{n(\alpha)}\}$ a sequence $\{\omega_{n(\alpha_k)}\}$ denoted by $\{\omega_k\}$ with $\int_A \omega_k(a)d\lambda \to \int_A \omega(a)d\lambda$. Let ξ_k

be the corresponding sequence extracted from $\xi_{n(\alpha)}$. Note that $\xi_k \equiv \xi_{n(\alpha_k)}$ is a sub-sequence of ξ_n. By Fact 12, we have a Gelfand integrable function $\xi : A \to X$ such that

$$\xi(a) \in Ls(\xi_k(a)) \subset Ls(\xi_{n(\alpha)}(a)) \text{ a.e,}$$

and

$$\int_A \xi(a) d\lambda \leq \int_A \omega(a) d\lambda.$$

Let $P = \{a \in A | \; \pi\omega(a) > 0\}$. Although the the elements of the space ba generally do not commute with the Gelfand integral, with the Pettis integral, however, they do. Note that $\int_A \omega(a)d\lambda$ can be seen as the Pettis integral. Then by Assumption (TE) and $\pi\mathbf{1} = 1$, we obtain that $\lambda(P) > 0$. The essence of the proof is contained in the next Lemma.

Lemma 2. $\xi(a) \prec_a \zeta$ implies that $\pi\omega(a) < \pi\zeta$ a.e on P.

Proof. See Appendix. ∎

Let $\pi = \pi_c + \pi_p$ be the Yosida-Hewitt decomposition and denote $\pi_c = \boldsymbol{p}$. The rest of the proof of Theorem 1 is completed exactly in the same way as Suzuki (2013a). Lemma 2 combined with the monotonicity (PR)(ii) imply that $\boldsymbol{p}\xi(a) \geq \boldsymbol{p}\omega(a)$ a.e. Then it follows from the resource feasibility condition that we have the budget conditions $\boldsymbol{p}\xi(a) = \boldsymbol{p}\omega(a)$ for almost all $a \in A$. The condition (E-1) follows immediately from Lemma 2 and Lemma 3 below.

Lemma 3. $\nu\left(\{a \in A | \; \pi\omega(a) = 0\}\right) = 0$.

Proof. See Appendix. ∎

Since $\int_A \xi^t(a)d\lambda \leq \int_A \omega^t(a)d\lambda \leq \tilde{\gamma} < \tilde{\beta}$ for each t, there exists a positive amount of consumers with $\xi^t(a) < \tilde{\beta}$. Then by the monotonicity (PR)(ii), one obtains that $p^t > 0$ for all t, hence $\int_A \xi(a)d\lambda = \int_A \omega(a)d\lambda$, or the condition (E-2) is met. This completes the proof of Theorem 1. ∎

6.2 Proofs of Theorems 2, 3 and Corollary.

First we show

Lemma 4. Let $(A, \mathcal{A}, \lambda)$ be an atomless measure space, $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ and $\xi : A \to X$ a measurable mapping. Define $\nu = (\xi, \mathcal{E})_* \lambda$. Then ξ is an equilibrium allocation of $\mathcal{E}$ if and only if ν is an equilibrium distribution of μ.

Proof. Suppose that ξ is an equilibrium allocation of $\mathcal{E}$. Then there exists a price vector $\boldsymbol{p}(\neq \mathbf{0}) \in \ell^\infty_+$ with $\lambda(E) = 1$ and $\int_A \xi(a)d\lambda = \int_A \omega(a)d\lambda$, where $E = \{a \in A | \; \boldsymbol{p}\xi(a) = \boldsymbol{p}\omega(a)$ and $\xi(a) \succsim_a \zeta$ whenever $\boldsymbol{p}\zeta \leq \boldsymbol{p}\omega(a)\}$. Let $F = \{(\xi, \succsim, \omega) \in X \times \mathcal{P} \times \Omega | \; \boldsymbol{p}\xi = \boldsymbol{p}\omega$ and $\xi \succsim \zeta$ whenever $\boldsymbol{p}\zeta \leq \boldsymbol{p}\omega\}$. Then $(\xi, \mathcal{E})(E) = F$, hence $\nu(F) = (\xi, \mathcal{E})_*\lambda(F) =$

$\lambda(E) = 1$, which proves the condition (D-1). Since $\xi_*\lambda = \nu_X$ and $\omega_*\lambda = \nu_\Omega = \mu_\Omega$, we have from the change of variable formula $\int_X \xi d\nu_X = \int_\Omega \omega d\mu_\Omega$. Hence the condition (D-2) is met. Finally, the condition (D-3) follows from $\nu_{\mathcal{P}\times\Omega} = \mathcal{E}_*\lambda = \mu$. The converse is also proved in a similar way. ∎

We now prove Theorem 2. Since $\mathcal{E}_*\lambda = \mu = \nu_{\mathcal{P}\times\Omega}$, we have from Fact 3 a measurable map ξ with $\nu = (\xi, \mathcal{E})_*\lambda$. Then ξ is an equilibrium allocation by Lemma 4. ∎

In order to prove that (a) implies (b) in Theorem 3, let ν be a symmetric equilibrium of an atomless economy μ, and $(A, \mathcal{A}, \lambda)$ an atomless probability space. Then there exists a measurable map $\sigma : \mathcal{P} \times \Omega \to X$ such that $\nu(Graph(\sigma)) = 1$. Suppose that $\mathcal{E} : A \to \mathcal{P} \times \Omega$ be a representation of μ. Set $\xi = \sigma \circ \mathcal{E}$. For each Borel set B of $X \times \mathcal{P} \times \Omega$, we have

$$\begin{aligned}\nu(B) &= \nu(B \cap Graph(\sigma)) = \nu(\{(\sigma(\succsim, \omega), \succsim, \omega) |\ (\sigma(\succsim, \omega), \succsim, \omega) \in B\}) \\ &= \mu(\{(\succsim, \omega) |\ (\sigma(\succsim, \omega), \succsim, \omega) \in B\} = \lambda(\{a \in A |\ (\xi(a), \mathcal{E}(a)) \in B\}) = (\xi, \mathcal{E})_*\lambda(B),\end{aligned}$$

hence $\nu = (\xi, \mathcal{E})_*\lambda$. That ξ is an equilibrium allocation follows from Lemma 4. Therefore (a) implies (b). Obviously, (b) $\Rightarrow$ (c) $\Rightarrow$ (d).

Now suppose that (d) holds. Since μ is atomless, it follows from Theorem 9.6.3 of Bogachev (2007) that there exists a representation $\mathcal{E}$ of μ on the Lebesgue space $(I, \mathcal{B}(I), \ell)$ which is one to one except for an ℓ-null set (almost one to one), where $I = [0, 1]$ and ℓ is the Lebesgue measure. Since $(I, \mathcal{B}(I), \ell)$ realizes ν, there exists an allocation ξ of $\mathcal{E}$ such that $\nu = (\xi, \mathcal{E})_*\ell$. It follows from Theorem 4.41 of Aliprantis and Border (2006) that $\xi = \sigma \circ \mathcal{E}$ for some measurable function σ from $\mathcal{P} \times \Omega$ to X. Then we have from $\nu = (\xi, \mathcal{E})_*\ell = \ell \circ (\xi, \mathcal{E})^{-1}$ that

$$\begin{aligned}\nu(Graph(\sigma)) &= \ell \circ (\xi, \mathcal{E})^{-1}(Graph(\sigma)) = \ell(\{a \in A |\ (\xi(a), \mathcal{E}(a)) \in Graph(\sigma)\}) \\ &= \ell(\{a \in A |\ (\sigma \circ \mathcal{E}(a), \mathcal{E}(a)) \in Graph(\sigma)\}) = \ell(\{a \in A |\ \mathcal{E}(a) \in \mathcal{P} \times \Omega\}) = 1,\end{aligned}$$

hence ν is symmetric. ∎

Finally we deduce Corollary. Let $(A, \mathcal{A}, \lambda)$ be saturated. Then it realizes any distributional equilibrium, hence symmetric equilibrium by Theorem 2. Conversely, let ν be non-symmetric distributional equilibrium of an economy μ. Then from Theorem 3, neither $(I, \mathcal{B}(I), \ell)$ nor $(I, \mathcal{I}, \bar{\ell})$ realizes ν. Since $(A, \mathcal{A}, \lambda)$ realizes ν, it follows from Fact 3 that $(A, \mathcal{A}, \lambda)$ is saturated. ∎

6.3 Proof of Theorem 4.

Let ϵ_n be a sequence of positive numbers decreasing to zero. As in Mas-Colell (1975) or Jones (1983), we can construct a sequence of finite subsets $K^n = \{t_1^n \dots t_{m_n}^n\}$ of K and a sequence of pairwise disjoint open sets B_i^n with $t_i^n \in B_i^n$ for $i = 1 \dots m_n$ such that denoting $B^n = \cup_{i=1}^{m_n} B_i^n$,

$$\begin{aligned}
&d(t_i^n, t) \le \epsilon_n \;\text{ for every }\; t \in B_i^n \;\text{ and for all }\; n, i = 1 \dots m_n,\\
&\int_A \omega(a) d\lambda(B^n) = \int_A \omega(a) d\lambda(K),\\
&K^n \subset K^{n+1} \;\text{ for all }\; n, \;\text{ and }\; K^n \to K \;\text{ in the topology of closed convergence.}
\end{aligned}$$

For each n, let $L^n = LS(t_1^n \dots t_{m_n}^n) \subset ca(K)$ be the linear space spanned by $\{\delta_{t_1^n} \dots \delta_{t_{m_n}^n}\}$ and set $X^n = X \cap L^n$. We then define $\mathcal{P}^n = \mathcal{P} \cap (X^n \times X^n)$. Let h^n be a map from $\mathcal{P}$ to $\mathcal{P}^n$ defined by $h^n(\succsim) = \succsim^n = \succsim \cap (X^n \times X^n)$.

Lemma 5. The map $h^n(\cdot)$ is continuous.

Proof. This is a proposition proved by Jones (1983), we reconstruct it in Appendix.

For each n, let $\psi^n : \mathcal{P} \times X \to \mathcal{P}^n \times X^n$ be a map defined by

$$\psi^n(\succsim, \omega) = (\succsim^n, \omega_n), \;\; \succsim^n = h^n(\succsim), \;\; \omega_n = \sum_{i=1}^{m_n} \omega(B_i^n) \delta_{t_i^n}.$$

It is obvious from definition that $\omega_n(K) = \omega(K)$. Since h^n is continuous and B_i^n are open, the map ψ^n is measurable. Set $\mathcal{E}^n = \psi^n \circ \mathcal{E} : A \to \mathcal{P}^n \times X^n$, $\mathcal{E}^n(a) = (\succsim_a^n, \omega_n(a))$. Then $\mathcal{E}^n$ is an economy with finite number of commodities. Since *support* $\left(\int_A \omega(a) d\lambda\right) = K$, we have by construction

$$\int_A \omega_n(a) d\lambda \gg \mathbf{0},$$

where we have identified X^n with $\mathbb{R}_+^{m_n}$. Then we have

Lemma 6. The economy $\mathcal{E}^n$ has an equilibrium, or there exist a price vector $\boldsymbol{q}_n \in \mathbb{R}_+^{m_n}$ with $\boldsymbol{q}_n \neq \mathbf{0}$ and an allocation $\xi_n : A \to X^n$ which satisfy

(E-1n) $\boldsymbol{q}_n \xi_n(a) \le \boldsymbol{q}_n \omega_n(a)$ and $\xi_n(a) \succsim_a \zeta$ whenever $\boldsymbol{q}_n \zeta \le \boldsymbol{q}_n \omega(a)$ a.e in A,

(E-2n) $\int_A \xi_n(a) d\lambda = \int_A \omega_n(a) d\lambda$.

Proof. Since $\int_A \omega_n(a) d\lambda \gg \mathbf{0}$ and the preferences are monotone, the assumptions for Theorem 2 in Hildenbrand (1974, p.151) are satisfied. ∎

Without loss of generality, we can assume $\|\boldsymbol{q}_n\| = sup\{\boldsymbol{q}_n(t) |\ t \in K^n\} = 1$ for all n. In the next lemma, the assumption of bounded marginal rate of substitution (RS) plays an essential role.

Lemma 7. Let $(\boldsymbol{q}_n, \xi_n)$ be the equilibrium obtained by Lemma 6. Then $(K^n, \boldsymbol{q}_n)$ are equi-continuous.

Proof. Appendix. ∎

It follows from Fact 8 that we can assume that $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ for some $\boldsymbol{q} \in C(K)$. Clearly $sup_{t\in K}\boldsymbol{q}(t) = 1$.

Lemma 8. Suppose that $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ and $\omega_n \to \omega \in X$ with $\boldsymbol{q}\omega > 0$. Assume that for each n, $\boldsymbol{q}_n\xi_n \le \boldsymbol{q}_n\omega_n$ and $\zeta_n \precsim^n \xi_n$ whenever $\boldsymbol{q}_n\zeta_n \le \boldsymbol{q}_n\omega_n$. Then if $\boldsymbol{q}(t^*) = 0$ for some $t^* \in K$, then $\xi_n(K) \to +\infty$.

Proof. Appendix. ∎

We now claim that there exists a $\beta > 0$ such that $\boldsymbol{q}_n(t) \ge \beta$ for all $t \in K^n$, $n = 1, 2 \ldots$. If this is not the case, there exists a sequence $\{t^n\} \subset K^n$ such that $t^n \to t^*$ for some $t^* \in K$ with $\boldsymbol{q}_n(t^n) \to 0$. Obviously this implies that $\boldsymbol{q}(t^*) = 0$ as well. Choose $s \in K$ such that $\boldsymbol{q}(s) = 1$ and take an open neighborhood U of s such that $\boldsymbol{q}(t) > 1/2$ for $t \in U$. Let $B = \{a \in A |\ \omega(a)(U) > 0\}$. Clearly B is measurable and by the assumption (AE), $\lambda(B) > 0$. Since $\boldsymbol{q}_n\omega_n(a) \to \boldsymbol{q}\omega(a) > 0$ a.e on B by Fact 7, it follows from Lemma 7 that $\int_A \omega_n(a)d\lambda(K) = \int_A \xi_n(a)d\lambda(K) \ge \int_B \xi_n(a)d\lambda(K) \to +\infty$. But this contradicts that $\int_A \omega_n(a)d\lambda(K) \to \int_A \omega(a)d\lambda(K) < +\infty$.

Take an $\epsilon > 0$. Since $sup_{t\in K}\boldsymbol{q}(t) = 1$ and $\boldsymbol{q}_n \ge \beta$, it follows that

$$0 \le \xi_n(a)(K) \le \frac{\omega(a)(K) + \epsilon}{\beta} \text{ for all } n \text{ large enough, a.e.}$$

Then applying Fact 12, we have shown that there exists a measurable map $\xi(a)$ such that

$$\int_A \xi(a)d\lambda \in Ls\left(\int_A \xi_n(a)d\lambda\right) \tag{1}$$

and

$$\xi(a) \in Ls(\xi_n(a)) \text{ a.e.} \tag{2}$$

We will show that $(\boldsymbol{q}, \xi)$ is an equilibrium for $\mathcal{E}$. Since $\int_A \omega_n(a)d\lambda = \int_A \xi_n(a)d\lambda$, $\boldsymbol{q}_n \to \boldsymbol{q}$ and $\omega_n(a) \to \omega(a)$ a.e, it follows from (1) and Fact 11 that

$$\int_A \omega(a)d\lambda = \lim_n \int_A \omega_n(a)d\lambda = \lim_n \int_A \xi_n(a)d\lambda = \int_A \xi(a)d\lambda,$$

and one obtains from (2) and Fact 7 that

$$\boldsymbol{q}\xi(a) \le \boldsymbol{q}\omega(a) \text{ a.e.}$$

Hence the condition (E-2) and the the budget conditions are met.

Suppose that there exists $\zeta \in X$ such that $\boldsymbol{q}\zeta \le \boldsymbol{q}\omega(a)$ and $\xi(a) \prec_a \zeta$. If $\boldsymbol{q}\omega(a) > 0$, then since the preferences are continuous, we can assume without loss of generality that $\boldsymbol{q}\zeta < \boldsymbol{q}\omega(a)$ and $\xi(a) \prec_a \zeta$. Let $\zeta_n = \sum_{i=1}^{m_n} \zeta(B_i^n)\delta_{t_i^n}$. Since $\boldsymbol{q}_n\omega_n(a) \to \boldsymbol{q}\omega(a)$ and $\boldsymbol{q}_n\zeta_n \to \boldsymbol{q}\zeta$, it follows from (E-1n) and (2) that $\boldsymbol{q}_n\zeta_n < \boldsymbol{q}_n\omega_n(a)$ and $\xi_n(a) \prec_a \zeta_n$ for n large enough, a contradiction. If $\boldsymbol{q}\omega(a) = 0$, then since $\boldsymbol{q}(t) \ge \beta$ for all $t \in K$, the budget set $\beta(a, \boldsymbol{q})$ is a singleton, or $\beta(a, \boldsymbol{q}) = \{\xi \in X |\ \boldsymbol{q}\xi \le 0\} = \{\boldsymbol{0}\}$, hence $\xi(a) = \boldsymbol{0}$ is trivially a maximal element in $\beta(a, \boldsymbol{q})$. Therefore the condition (E-1) is met and we complete the proof. ∎

7 Historical Notes

In what follows, we will give some historical remarks on the studies of saturated measure spaces and their applications to game theory and equilibrium theory.

The saturated measure spaces have been grown out from the study of the structures of measure algebras. Following classical isomorphism theorem is due to Caratheodory (1939) and Halmos-Neumann (1942). For an example of modern textbook-level expositions, see Royden (1988, p.399).

Isomorphism Theorem. Every separable measure algebra associated with an atomless probability space is isomorphic to the measure algebra associated with the Lebesgue space.

Maharam (1942 and 50) extended this classical result and established the fundamental structure theorem of measure algebras (Fajardo-Keisler (2002, Theorem 3B.6), see also Fremlin (2002, 332B)).

Structure Theorem. Every measure algebra associated with an atomless probability measure space is isomorphic to the (finite or countably infinite) convex combination of the algebras $[0,1]^{\mathbf{m_i}}$, $i = 1, 2 \ldots$.

In the structure theorem, $\mathbf{m_i}$ are arbitrary infinite cardinals and the set of cardinals $\{\mathbf{m_1}, \mathbf{m_2}, \ldots\}$ is unique. The set is called the Maharam spectrum of the algebra. If the measure algebra is homogeneous, the spectrum is a singleton $\{\mathbf{m}\}$ and the cardinal $\mathbf{m}$ is the Maharam type of the algebra. Therefore the structure theorem asserts that the homogeneous and atomless (probability) measure algebras are completely characterized as their isomorphic classes by the Maharam types.

Note that the isomorphism theorem corresponds to the case of a homogeneous algebra where $\mathbf{m}$ is countable. Since a separable algebra has a countable and dense subset, every subalgebra is countably generated, or it is homogeneous with its Maharam type of the countable cardinal. Lebesgue space is also homogeneous and its Maharam type is countable (Fremlin (2002, 331X, p.130)). Hence by the structure theorem the both algebras belong to the same isomorphism class of the measure algebras with the Maharam type of the countable (infinite) cardinal (represented by $[0,1]^{\mathbb{N}}$). This is nothing but the statement of the isomorphism theorem.

Based on the works of Maharam (1942 and 50), Hoover-Keisler (1984) defined the saturated measure space (they called it the $\aleph_1$ atomless measure space), and proved its equivalence with the saturated property (Fact 3 (b) in Section 2.1) in their study of the stochastic process (the existence of strong solutions for stochastic integral equations). They also observed that the atomless Loeb space introduced by Loeb (1975) is a special case of the saturated measure space. By the subsequent works (Keisler-Sun (2009), Sun-Yannelis (2008) and Loeb-Sun (2009); among others), it has become apparent that results on Loeb spaces can be transferred straightforwardly to saturated measure spaces.

Fajardo-Keisler (2002) elaborated on the Maharam spectra of the saturated measure spaces. Definition 1 is essentially due to them. They also proved (Fajardo-Keisler (2002, Theorem 3B.7))

Theorem. A probability space is saturated if and only if its Maharam spectrum is a set of uncountable cardinals.

The very comprehensive and systematic expositions for all these results and observations can be found in the monumental monograph of Fremlin (2002).

The most impressive and far-reaching results in the applications of saturated measure spaces for mathematical analysis are evidently those for the distributions of the correspondences due to Sun (1996) and Keisler-Sun (2009). Let $(A, \mathcal{A}, \lambda)$ be a saturated measure space and F be a correspondence from A to a complete separable metric space X. The distribution of F denoted by $\mathcal{D}_F$ is the set of all distributions of the measurable selections of F, $\mathcal{D}_F = \{f_*\lambda | \ f$ is a measurable selection of $F\}$. The next theorem was proved by Sun (1996) for the atomless Loeb space, and Keisler-Sun (2009) for a general saturated space.

Theorem. For any correspondence F, $\mathcal{D}_F$ is convex.

They also proved that $\mathcal{D}_F$ is closed and compact (in the weak* topology of probability measures) if F is closed valued and convex valued respectively. Moreover the following is true. For a metric space Y, let G be a correspondence from $A \times Y$ to X with $G(a, y) \subset F(a)$ for a compact valued correspondence F a.e, $G(\cdot, y)$ is measurable for all y and $G(a, \cdot)$ is upper-hemi continuous for almost all a. Then $\mathcal{H}(y) \equiv \mathcal{D}_{G(\cdot,y)}$ is upper-hemi continuous.

The next theorem which was also proved by Sun (1996) for atomless Loeb spaces and by Keisler-Sun (2009) for saturated spaces roughly says that for any "randomized solution", there exists a corresponding "purified (non-random) solution" (see also Loeb-Sun (2009) and Podczeck (2009)).

Theorem. Let Φ be a measurable mapping from A to the space $\mathcal{M}(X)$ of probability measures on X. Then there exists a measurable mapping ϕ from A to X such that (a) for every Borel set B in X, $\phi_*\lambda(B) = \int_A \Phi(a)(B)d\lambda$, and (b) $\phi(a) \in \mathit{support}(\Phi(a))$ a.e.

Keisler-Sun (2009) showed that all these results failed for every atomless probability space which is not saturated. Hence the saturation is also necessary for each of these theorems to hold.

As seen in the present paper, the theory of integrations for maps and correspondences defined on a saturated measure space which take their values in Banach spaces is important in game theory and equilibrium theory. The study of these integrals started from Sun (1997) for atomless Loeb spaces, and subsequently Podczeck (2008) and Sun-Yannelis (2008) for general saturated spaces. They extended the classical results of Aumann (1965) and Richter (1963) which assert that for an integrably bounded correspondence from an atomless measure space to a finite dimensional euclidean space its integral is compact and convex, to Banach space valued correspondences.

Podczeck (2008) in particular observed a remarkable fact that one can extend the Lebesgue space to a saturated measure space (without any use of the nonstandard technique) by "enriching" the σ-algebra. Hence for a saturated measure space, the space itself does not necessarily have an extraordinary large cardinality. As Podczeck stressed, this has an important implication when a measure space of agents for game theory or economic theory is assumed to be saturated as in the present paper (this point has been already addressed in Section 1).

The results including the Fatou's lemmas of Khan-Sagara (2013 and 2014) and Khan-Sagara-Suzuki (2014) were obtained exactly on this line of research. Studies of the Fatou's lemma have their own history and it is too long to be presented here. See the introductory sections of Khan-Sagara (2014) and Khan-Sagara-Suzuki (2014) and the references in them. Here we only cite Sun (1997) and Loeb-Sun (2007) for the exact versions of the lemma, compared to approximate versions (see Section 1) of Cornet-Medecin (2002), Balder-Sambucini (2005) and Suzuki (2013c), among others.

The games with a continuum of players (the large games) were introduced by Schmeidler (1973). He proved the existence of strategic (individual) Nash equilibria when the reactions of players in the payoff functions are contained as the integral of the strategy profiles. On the other hand, Mas-Colell (1984) defined a distributional form of the large game. The set of players' strategies (action set) X is assumed to be a compact metric space and the payoff function is a continuous function on $X \times \mathcal{M}(X)$, where a measure $\mu \in \mathcal{M}(X)$ means the distribution of actions taken by the players of the game. The (distributional) game $\mathcal{G}$ is defined to be a probability measure on the set of payoff functions $\mathcal{U}(X \times \mathcal{M}(X))$, and the Nash equilibrium is a probability measure ν on $X \times \mathcal{U}(X \times \mathcal{M}(X))$, an element of $\mathcal{M}(X \times \mathcal{U}(X \times \mathcal{M}(X)))$. Mas-Colell proved the existence of a Nash equilibrium distribution in a very simple and elegant manner.

We can now define the strategic (individual) game to be a measurable map Υ from a measure space $(A, \mathcal{A}, \lambda)$ to $\mathcal{U}(X \times \mathcal{M}(X))$, $a \mapsto \Upsilon(a) = u(a; \xi, \mu)$. The strategy profile (and the Nash equilibrium) is defined to be a measurable map $\xi : A \to X$. Rath (1992) gave a simple proof for the existence of Nash equilibria for a case that X is a compact subset of $\mathbb{R}^\ell$ and the second variable of $u(a, \cdot, \cdot)$ is not a measure μ but an integral $\int_A \xi(a) d\lambda$ (as in Schmeidler (1973)).

In Section 1, we pointed out that the condition on a measure space postulated by Podczeck (1997) contains essential ingredients of the saturation. Indeed, Podczeck (1997) showed that his condition on the measure space implied the convexity of the integral (Liapounoff's theorem) and an exact version of Fatou's lemma. This observation was also verified by Noguchi (2009) in which he proved the existence of Nash equilibria for a strategic large game $\Upsilon : A \to \mathcal{U}(X \times \mathcal{M}(X))$, where X is a compact metric space. In order to prove this, he showed that the "Podczeck condition" admits essentially the saturation property (Definition 2 in Section 2.1), hence we can go "back and forth" between the distributional equilibria and the strategic equilibria. More precisely, we have a distributional equilibrium $\nu \in \mathcal{M}(X \times \mathcal{U}(X \times \mathcal{M}(X)))$ for the distributional game $\mathcal{G} \equiv \Upsilon_* \lambda$ which has been proven to exist by Mas-Colell (1984), and apply the saturated property to obtain a strategic equilibrium $\xi : A \to X$ satisfying $(\xi, \Upsilon)_* \lambda = \nu$.

Fully armed with the concept of the saturated measure space, Carmona and Podczeck (2009) showed the equivalence between the existence of distributional and strategic equilibria, and discussed the results previously obtained by Schmeidler (1973), Rashid (1983), Mas-Colell (1984), Khan and Sun (1999) and Podczeck (2009) in a systematic and unified manner.

Rath-Sun-Yamashige (1995) and Khan-Rath-Sun (1997) constructed examples of strategic games which do not have any Nash equilibria. These counter examples culminated in the next fundamental theorem which shows that the saturation is necessary and sufficient for the existence of Nash equilibria (Keisler-Sun (2009), see also Sun-Zhang (2013)).

Theorem. Let $(A, \mathcal{A}, \lambda)$ be an atomless probability space, and X an uncountable compact metric space. Then $(A, \mathcal{A}, \lambda)$ is saturated if and only if every game $\Upsilon : A \to \mathcal{U}(X \times \mathcal{M}(X))$ has a Nash equilibrium.

The applications of the saturated measure space to general equilibrium theory are relatively few. Lee (2013) proved a Gale-Nikaido lemma to prove the existence of competitive equilibria for an exchange economy with a separable Banach space of the commodities with an interior point in the positive orthant and the saturated measure space of the consumers. Suzuki (2014) discussed the competitive equilibrium for a production economy with infinitely many indivisible commodities. He proved an equilibrium existence theorem for a distributional form of the economy as in Suzuki (2013a and b), and used the "back and forth" argument to show the existence of equilibrium for an economy of the individual form. The power of the saturated spaces is particularly manifest in the models with indivisible commodities, since one can not assume the convexity of preferences for such models.

Appendix

Lemma 2. $\xi(a) \prec_a \zeta$ implies that $\pi\omega(a) < \pi\zeta$ a.e on P.

Proof. If the Lemma was false, there exists $\zeta(a) = (\zeta^t(a)) \in X$ such that $\pi\zeta(a) \leq \pi\omega(a)$ and $\xi(a) \prec_a \zeta(a)$ on a subset of P with μ-positive measure. We can assume without loss of generality that $\pi\zeta(a) < \pi\omega(a)$ and $\xi(a) \prec_a \zeta(a)$. Let $\zeta_n(a) = (\zeta^1(a) \ldots \zeta^n(a), 0, 0 \ldots)$ be the projection of $\zeta(a)$ to X^n. We have for sufficiently large N that $\pi\zeta_N(a) \leq \pi\zeta(a) < \pi\omega(a)$ and $\xi(a) \prec_a \zeta_N(a)$. We now show that for every α, there exists an $\alpha_0 \geq \alpha$ such that $\xi_{n(\alpha_0)}(a) \prec_a \zeta_N(a)$ a.e. If not, we have for some α_0, $\xi_{n(\alpha)}(a) \succsim_a \zeta_N(a)$ for all $\alpha \geq \alpha_0$. Since $\xi(a) \in Ls(\xi_{n(\alpha)}(a))$ a.e, we have $\xi(a) \succsim_a \zeta_N(a)$ a.e, a contradiction.

Then it follows from $\pi_{n(\alpha)} \to \pi$ that for some α_0 with $n(\alpha_0) \equiv n_0 \geq N$, $0 \leq \pi_{n_0}\zeta_N(a) < \pi_{n_0}\omega(a) = \pi_{n_0}\omega_{n_0}(a)$, and $\xi_{n_0}(a) \prec_a \zeta_N(a)$, or $\xi_{n_0}(a) \prec_a^{n_0} \zeta_N(a)$. This contradicts the fact that $(\pi_{n_0}, \xi_{n_0}(a))$ is a quasi-equilibrium for $\mathcal{E}^{n_0}$. ∎

Lemma 3. $\nu\left(\{a \in A |\ \pi\omega(a) = 0\}\right) = 0$.

Proof. Let $Q = \{a \in A |\ \pi\omega(a) = 0\}$ and assume that $\lambda(Q) > 0$. By the assumption (IR), there exists an allocation ϕ with $\int_Q (\omega(a) - \phi(a)) d\lambda + \int_P \xi(a) d\lambda \in \int_P \{\zeta \in X |\ \xi(a) \prec_a \zeta\} d\lambda$.

Since $\boldsymbol{p}\xi(a) = \boldsymbol{p}\omega(a) \leq \|\boldsymbol{p}\|\tilde{\gamma} < \|\boldsymbol{p}\|\tilde{\beta}$ a.e on P, one has $\int_P \{\zeta \in X | \ \xi(a) \prec_a \zeta\} d\lambda \neq \emptyset$. Then there exists a map ζ on P to X such that $\xi(a) \prec_a \zeta(a)$ a.e on P with $\int_P \zeta(a) d\lambda = \int_Q (\omega(a) - \phi(a)) d\lambda + \int_P \xi(a) d\lambda$. Then $\boldsymbol{p}\omega(a) < \boldsymbol{p}\zeta(a)$ a.e on P and $0 \leq \boldsymbol{p} \int_Q \omega(a) d\lambda = \int_Q \boldsymbol{p}\omega(a) d\lambda \leq \int_Q \pi\omega(a) d\lambda \leq 0$, hence

$$\begin{aligned} \boldsymbol{p} \int_P \xi(a) d\lambda &= \boldsymbol{p} \int_P \omega(a) d\lambda < \boldsymbol{p} \int_P \zeta(a) d\lambda \\ &= \boldsymbol{p} \int_Q (\omega(a) - \phi(a)) d\lambda + \boldsymbol{p} \int_P \xi(a) d\lambda \leq \boldsymbol{p} \int_P \xi(a) d\lambda, \end{aligned}$$

a contradiction. ∎

Lemma 5. The map $h^n(\cdot)$ is continuous.

Proof. Take a sequence $\{\succsim^k\}$ in $\mathcal{P}$ such that $\succsim^k \to \succsim \in \mathcal{P}$, and let $H_k = h^n(\succsim^k)$ and $H = h^n(\succsim)$. Recall that the topology of closed convergence τ_c is generated by the base

$$[K; G_1 \dots G_m] = \{F | \ F \cap K = \emptyset, F \cap G_i \neq \emptyset, i = \dots m\},$$

where K compact and G_i are open in $X \times X$, $i = 1 \dots m$.

If K is a compact subset of $X^n \times X^n$ with $H \cap K = \emptyset$, then K is also a compact subset in $X \times X$. Hence for large k, $H_k \cap K = \emptyset$ as well. Suppose G is an open set of $X^n \times X^n$ with $H \cap G \neq \emptyset$. Let $(\xi_1, \xi_2) \in H \cap G$. Since G is open, it follows from (ii) of Assumption (PR) that we can assume that $\xi_2 \prec \xi_1$. Then $K = \{(\xi_2, \xi_1)\}$ is a compact subset of $X^n \times X^n$ with $H \cap K = \emptyset$. Then $H_k \cap K = \emptyset$ for k large enough. Hence $(\xi_1, \xi_2) \in H_k$ by completeness of preferences. This shows that $H_k \to H$ as desired. ∎

Lemma 7. Let $(\boldsymbol{q}_n, \xi_n)$ be the equilibrium obtained by Lemma 8. Then $(K^n, \boldsymbol{q}_n)$ are equi-continuous.

Proof. Suppose that $(K^n, \boldsymbol{q}_n)$ are not equi-continuous. Then we can assume that there exist sequences (t_n), (s_n) such that

$$d(t_n, s_n) \to 0 \ \text{ and } \ \lim_{n \to \infty} \frac{\boldsymbol{q}_n(t_n)}{\boldsymbol{q}_n(s_n)} > 1.$$

For t_n with $\xi^n(\{t_n\}) > 0$, define

$$\xi_n^{t_n, s_n} = \xi_n - \xi_n(\{t_n\})\delta_{t_n} + \left(\frac{\boldsymbol{q}_n(t_n)}{\boldsymbol{q}_n(s_n)}\right) \xi_n(\{t_n\})\delta_{s_n}.$$

It follows from $\int_A \xi_n(a) d\mu = \int_A \omega_n(a) d\mu \gg \mathbf{0}$ that $\mu(\{a \in A | \ \xi_n(a)(t_n) > 0\}) > 0$. If $\xi_n(a)(t_n) > 0$, then by the assumption (RS), we have for n sufficiently large that $\xi_n(a) \prec_a \xi_n^{t_n, s_n}(a)$. This is a contradiction, since $\boldsymbol{q}_n \xi_n^{t_n, s_n}(a) = \boldsymbol{q}_n \xi_n(a)$. ∎

Lemma 8. Suppose that $(K^n, \boldsymbol{q}_n) \to (K, \boldsymbol{q})$ and $\omega_n \to \omega \in \Omega$ with $\boldsymbol{q}\omega > 0$. Assume that for each n, $\boldsymbol{q}_n\xi_n \leq \boldsymbol{q}_n\omega_n$ and $\zeta_n \precsim^n \xi_n$ whenever $\boldsymbol{q}_n\zeta_n \leq \boldsymbol{q}_n\omega_n$. Then if $\boldsymbol{q}(t^*) = 0$ for some $t^* \in K$, then $\xi_n(K) \to +\infty$.

Proof Suppose not. Then by taking subsequence if necessary, we can assume that $\xi_n \to \xi$ for some $\xi \in X$. We now claim that $\boldsymbol{q}\xi \leq \boldsymbol{q}\omega$ and $\zeta \precsim \xi$ whenever $\boldsymbol{q}\zeta \leq \boldsymbol{q}\omega$. To see this, suppose that there exists $\zeta \in X$ with $\boldsymbol{q}\zeta \leq \boldsymbol{q}\omega$ and $\xi \prec \zeta$. Since $\boldsymbol{q}\omega > 0$ we can assume without loss of generality that $\boldsymbol{q}\zeta < \boldsymbol{q}\omega$ and $\xi \prec \zeta$. Setting $\zeta_n = \sum_{i=1}^{m_n} \zeta(B_i^n)\delta_{t_i^n}$, we have $\zeta_n \to \zeta$, hence $\boldsymbol{q}_n\zeta_n < \boldsymbol{q}_n\omega_n$ and $\zeta_n \succsim^n \xi_n$ for n large enough, contradicting the assumption. This can not be the case, however, since $\boldsymbol{q}(\xi + \delta_{t^*}) = \boldsymbol{q}\xi$ and $\xi \prec \xi + \delta_{t^*}$. ∎

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