# Consistency, anonymity, and the core on the domain of convex games*

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**Abstract**

Peleg (1986) and Tadenuma (1992) provide two well-known axiomatic characterizations of the core on the domain of balanced TU games. Peleg's characterization says that the core is the only solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, and a reduced game property introduced by Davis and Maschler (1965). Tadenuma's characterization says that the core is the only solution that satisfies *non-emptiness*, *individual rationality* and a reduced game property introduced by Moulin (1985). In this note, we investigate what happens when the domain is restricted to the class of convex TU games. In particular, we show that (i) the core is not the only solution that satisfies Peleg's four axioms and *anonymity*; (ii) the core is the only solution that satisfies Peleg's four axioms, *anonymity*, and additional three axioms; and (iii) the core is not the only solution that satisfies Tadenuma's three axioms and *anonymity*.

## 1 Introduction

The core (Gillies, 1959) is one of the most important solutions for cooperative games. It is important mainly because it satisfies many desirable

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properties. In particular, it satisfies two kinds of reduced game properties, namely, "max consistency" (Davis and Maschler, 1965) and "complement consistency" (Moulin, 1985).[1] There are two well-known axiomatic characterizations of the core on the domain of balanced TU games based on each of these two axioms: (i) the core is the only solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, and *max consistency* (Peleg, 1986); (ii) it is the only solution that satisfies *non-emptiness*, *individual rationality* and *complement consistency* (Tadenuma, 1992).[2]

In this note, we investigate what happens when the domain is restricted to the class of convex TU games. Although the core satisfies Peleg's four axioms on this domain, it is not the only one.[3] It so happens that except for the core itself, all existing examples of such solutions violate *anonymity*. So, one may conjecture that an axiomatic characterization of the core might be obtained by adding *anonymity* to Peleg's three axioms. In this note, we disprove this conjecture. We also consider a similar problem for *complement consistency*. In particular, we show that the core is not the only solution on the domain of convex games that satisfies Tadenuma's three axioms and *anonymity*.

# 2 Definitions and questions

Let $\mathcal{N}$ denote the class of non-empty and finite subsets of the set $\mathbb{N}$ of natural numbers. We use $\subset$ for strict set inclusion, and $\subseteq$ for weak set inclusion. There is an infinite set of "potential" players indexed by the members of $\mathbb{N}$. Given $N \in \mathcal{N}$, a **transferable utility (TU) game for $N$** is a function $v\colon 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. A game $v$ for $N$ is **convex** (Shapley, 1971) if for all $i \in N$ and all $S, T \in 2^N$ with $i \in S \subset T$, we have $v(S) - v(S \setminus \{i\}) \leq v(T) - v(T \setminus \{i\})$. Let $\mathcal{V}^N_{vex}$ denote the class of convex games for $N$. A game $v$ is **balanced** if for all nonnegative-valued function $\delta\colon 2^N \to \mathbb{R}_+$ such that for all $i \in N$, $\sum_{S \ni i} \delta(S) = 1$, we have $v(N) \geq \sum_{S \in 2^N} \delta(S) v(S)$.

[1] These two axioms are usually called "DM-consistency" and "M-consistency", respectively. We use the terminology introduced by Thomson (1996) and call them max consistency and complement consistency because each name suggests how the underlying "reduced games" are defined in each case.

[2] Voorneveld and van den Nouweland (1998) provide an axiomatization of the core which is closely related to Tadenuma's result.

[3] Although this fact is widely known, we don't know any published or unpublished paper that mentions it.

Given a game $v$ for $N$, the **core of $v$**, denoted $C(v)$, is the set of vectors $x \in \mathbb{R}^N$ such that $\sum_{i \in N} x_i = v(N)$ and for all $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$. It is well-known that a game is balanced if and only if its core is non-empty (Bondareva, 1963; Shapley, 1967). It is also well-known that every convex game is balanced (Shapley, 1971).

Suppose that for all $N \in \mathcal{N}$, a class $\mathcal{V}^N$ of games for $N$ is specified, and let $\mathcal{V} \equiv \bigcup_{N \in \mathcal{N}} \mathcal{V}^N$. A **solution on $\mathcal{V}$** is a mapping that assigns to all $N \in \mathcal{N}$ and all $v \in \mathcal{V}^N$ a set of vectors $x \in \mathbb{R}^N$ with $\sum_{i \in N} x_i \leq v(N)$. The core, as a mapping, is a solution on the class of balanced games. We use $\varphi$ as a generic notation for solutions.

Next, we define *max consistency* (Davis and Maschler, 1965) and *complement consistency* (Moulin, 1985). Each of these axioms says that the original choice should be "confirmed" in associated "reduced games," obtained by imagining a subset of players leaving with their payoffs and reevaluating the situation from the viewpoint of the remaining players. The different definitions come from the various ways of performing this reassessment.

Given $N \in \mathcal{N}$, a game $v$ for $N$, $x \in \mathbb{R}^N$, and $N' \subset N$, the **max reduced game of $v$ relative to $x$ and $N'$**, denoted $\hat{r}^x_{N'}(v)$, is defined by setting for all $S \subseteq N'$,

$$\hat{r}^x_{N'}(v)(S) \equiv \begin{cases} \max_{T \subseteq N \setminus N'} \left[ v\big(S \cup T\big) - \sum_{i \in T} x_i \right] & \text{if } S \notin \{N', \emptyset\}, \\ v(N) - \sum_{i \in N \setminus N'} x_i & \text{if } S = N', \\ 0 & \text{if } S = \emptyset. \end{cases}$$

**Max consistency**: A solution $\varphi$ satisfies *max consistency* if and only if for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $x \in \varphi(v)$, and all $N' \subset N$, we have $\hat{r}^x_{N'}(v) \in \mathcal{V}^{N'}$ and $x_{N'} \in \varphi(\hat{r}^x_{N'}(v))$.

Given $N \in \mathcal{N}$, a game $v$ for $N$, $x \in \mathbb{R}^N$, and $N' \subset N$, the **complement reduced game of $v$ relative to $x$ and $N'$**, denoted $r^x_{N'}(v)$, is defined by setting for all $S \subseteq N'$,

$$r^x_{N'}(v)(S) \equiv \begin{cases} v\big(S \cup (N \setminus N')\big) - \sum_{i \in N \setminus N'} x_i & \text{if } S \neq \emptyset, \\ 0 & \text{if } S = \emptyset. \end{cases}$$

**Complement consistency**: A solution $\varphi$ satisfies *complement consistency* if and only if for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N$, all $x \in \varphi(v)$, and all $N' \subset N$, we have $r^x_{N'}(v) \in \mathcal{V}^{N'}$ and $x_{N'} \in \varphi(r^x_{N'}(v))$.

The following axioms apply to games with a fixed set of players.

**Non-emptiness**: For all $v \in \mathcal{V}^N$, $\varphi(v) \neq \emptyset$.

**Individual rationality**: For all $v \in \mathcal{V}^N$, all $x \in \varphi(v)$, and all $i \in N$, we have $x_i \geq v(\{i\})$.

**Super-additivity**: For all $v, w \in \mathcal{V}^N$ with $v + w \in \mathcal{V}^N$, we have $\varphi(v) + \varphi(w) \subseteq \varphi(v + w)$.

As mentioned above, on the domain of balanced games, (i) the core is the only solution satisfying *non-emptiness*, *individual rationality*, *super-additivity*, and *max consistency* (Peleg, 1986); and (ii) the core is the only solution satisfying *non-emptiness*, *individual rationality* and *complement consistency* (Tadenuma, 1992). On the domain of convex games, the core satisfies *max consistency* (Maschler, Peleg, and Shapley, 1972), as well as *non-emptiness*, *individual rationality*, and *super-additivity.* Given a strict ordering $\prec$ on $\mathbb{N}$, consider the following solution $\varphi^{\prec}$, which picks for each convex game the "marginal contribution vector" with respect to $\prec$: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N_{vex}$, and all $i \in N$,

$$\varphi^{\prec}_i(v) \equiv v\left(\{j \in N \mid j \prec i\} \cup \{i\}\right) - v\left(\{j \in N \mid j \prec i\}\right).$$

On the domain of convex games, this solution satisfies *max consistency* (Orshan, 1994; Núñez and Rafels, 1998; Hokari, 2005). Moreover, it satisfies *non-emptiness*, *super-additivity*, and *individual rationality.* This means that on the domain of convex games, the core is **not** the only solution that satisfies Peleg's four axioms. Clearly, the above solution violates the following axiom:

**Anonymity**: For all $N, N' \in \mathcal{N}$ with $|N| = |N'|$, all $v \in \mathcal{V}^N$, and all $w \in \mathcal{V}^{N'}$, if there exists a bijection $\pi\colon N \to N'$ such that for all $S \subseteq N$, $w(\{\pi(i)\}_{i \in S}) = v(S)$, then for all $x \in \varphi(v)$, we have $(x_{\pi^{-1}(j)})_{j \in N'} \in \varphi(w)$.

As far as we know, other than the core itself, no *anonymous* solution on the domain of convex games that satisfies Peleg's three axioms can been found in the literature. So, the first question we would like to ask is the following:

**Question 1.** On the domain of convex games, is the core the only solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, *max consistency*, and *anonymity*?

Now, let us consider *complement consistency.* It is trivial to show that the core satisfies this axiom on the domain of convex games. Given that the core satisfies this property on the domain of balanced games, the only thing we have to check is whether the class of convex games is closed under the reduction operation underlying *complement consistency*.

**Lemma 1.** On the domain of convex games, the core satisfies *complement consistency*.

*Proof.* Let $N \in \mathcal{N}$, $v \in \mathcal{V}_{vex}^N$, $x \in C(v)$, and $N' \in \mathcal{N}$ with $N' \subset N$. Since the core satisfies *complement consistency* on the domain of balanced games (Tadenuma, 1992), it is enough to show that $r_{N'}^x(v)$ is convex.

Let $i \in N'$ and $S, T \in 2^{N'}$ be such that $i \in S \subset T$. If $|S| \geq 2$, then

$$\begin{aligned} & r_{N'}^x(v)(T) - r_{N'}^x(v)(T \setminus \{i\}) - r_{N'}^x(v)(S) + r_{N'}^x(v)(S \setminus \{i\}) \\ = \;& v(T \cup (N \setminus N')) - v((T \setminus \{i\}) \cup (N \setminus N')) \\ & -v(S \cup (N \setminus N')) + v((S \setminus \{i\}) \cup (N \setminus N')) \\ \geq \;& 0, \end{aligned}$$

where the last inequality follows from the convexity of $v$.

If $|S| = \{i\}$, then

$$\begin{aligned} & r_{N'}^x(v)(T) - r_{N'}^x(v)(T \setminus \{i\}) - r_{N'}^x(v)(\{i\}) \\ = \;& v(T \cup (N \setminus N')) - v((T \setminus \{i\}) \cup (N \setminus N')) - v(\{i\} \cup (N \setminus N')) + \sum_{j \in N \setminus N'} x_j \\ \geq \;& v(T \cup (N \setminus N')) - v((T \setminus \{i\}) \cup (N \setminus N')) - v(\{i\} \cup (N \setminus N')) + v(N \setminus N') \\ \geq \;& 0, \end{aligned}$$

where the last two inequalities follow from $x \in C(v)$ and the convexity of $v$, respectively. □

So, the next question is:

**Question 2.** On the domain of convex games, is the core the only solution that satisfies *non-emptiness*, *individual rationality*, and *complement consistency*? If it is not, what if *anonymity* is added to this list of axioms?

# 3 Results

Consider the following solution $\varphi^*$ on the domain of convex games: for all $N \in \mathcal{N}$ and all $v \in \mathcal{V}_{vex}^N$, let

$$\mathcal{S}(v) \equiv \left\{ S \in 2^N \setminus \{N, \emptyset\} \,\middle|\, \forall x \in C(v),\ \sum_{i \in S} x_i = v(S) \right\},$$

and

$$\varphi^*(v) \quad \equiv \quad \left\{ x \in C(v) \,\middle|\, \forall S \notin \mathcal{S}(v),\ \sum_{i \in S} x_i > v(S) \right\}.$$

Non-emptiness of $\varphi^*(v)$ can be checked as follows: if $S \notin \mathcal{S}(v)$, then there exists $y^S \in C(v)$ such that $\sum_{i \in S} y_i^S > v(S)$. Since $C(v)$ is a convex set, one can obtain an element of $\varphi^*(v)$ by taking a strict convex combination of these $y^S$'s. Essentially, $\varphi^*$ is the relative interior of the core. Note that $\varphi^*$ trivially satisfies *individual rationality* and *anonymity.* On the domain of balanced games, $\varphi^*$ satisfies *max consistency* (Yanovskaya, 1999). Together with the fact that the core satisfies the property on the domain of convex games, the *max consistency* of $\varphi^*$ on the domain of balanced games implies the *max consistency* of $\varphi^*$ on the domain of convex games. We show that it also satisfies *super-additivity.*

**Lemma 2.** On the domain of convex games, $\varphi^*$ satisfies *super-additivity.*

*Proof.* Let $N \in \mathcal{N}$, $v, w \in \mathcal{V}^N_{vex}$, $x \in \varphi^*(v)$, and $y \in \varphi^*(w)$. Since the core is *super-additive* and $\varphi^*$ is a subsolution of the core, we have $x + y \in C(v + w)$.

Let $S \notin \mathcal{S}(v + w)$. Then there exists $z \in C(v + w)$ such that $\sum_{i \in S} z_i > v(S) + w(S)$. Note that on the domain of convex games, the core is *additive* (Dragan, Potters, and Tijs, 1989).[4] Thus, there exist $x' \in C(v)$ and $y' \in C(w)$ such that $z = x' + y'$. Thus, $\sum_{i \in S} x'_i + \sum_{i \in S} y'_i = \sum_{i \in S} z_i > v(S) + w(S)$ so that either $S \notin \mathcal{S}(v)$ or $S \notin \mathcal{S}(w)$. Since $x \in \varphi^*(v)$ and $y \in \varphi^*(w)$, either $\sum_{i \in S} x_i > v(S)$ or $\sum_{i \in S} y_i > w(S)$. Together with $x \in C(v)$ and $y \in C(w)$, this implies $\sum_{i \in S} x_i + \sum_{i \in S} y_i > v(S) + w(S)$. □

We have the following result:

**Proposition 1.** On the domain of convex games, the core is **not** the only solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, *max consistency*, and *anonymity.*

Next, let us consider Question 2. Our starting point is the solution $\varphi^{\prec}$, defined in Section 2, that picks for each game the marginal contribution vector with respect to a given ordering $\prec$ of players. Although $\varphi^{\prec}$ itself does not satisfy *complement consistency*, we can enlarge it so that the resulting

[4]The definition of *additivity* is obtained by replacing $\subseteq$ with $=$ in the definition of *super-additivity.*

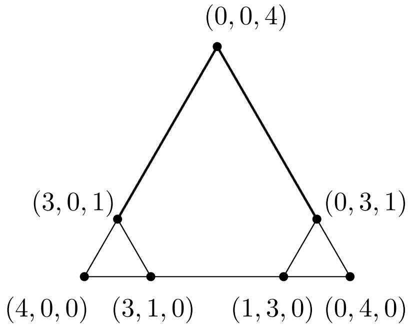


Figure 1: Let $N \equiv \{1,2,3\}$ and $v \in \mathcal{V}^N_{vex}$ be such that $v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$, $v(\{1,2\}) = 0$, $v(\{1,3\}) = v(\{2,3\}) = 1$, and $v(N) = 4$. Then, $v(N) - v(\{2,3\}) = v(N) - v(\{1,3\}) < v(N) - v(\{1,2\})$. So, there are two strict orderings on $N$ that satisfy condition (i) in the definition of $\varphi^{**}(v)$: $1 \prec 2 \prec 3$ and $2 \prec' 1 \prec' 3$. Thus, $\varphi^{**}(v) = \big\{x \in C(v) \bigm| x_2 = 0 \text{ or } x_1 = 0\big\}$.

solution satisfies the axiom. Then we endogenize the strict ordering $\prec$ to make the resulting solution *anonymous*.

Consider the following solution $\varphi^{**}$ on the domain of convex games: for all $N \in \mathcal{N}$, all $v \in \mathcal{V}^N_{vex}$, and all $x \in C(v)$, $x \in \varphi^{**}(v)$ if and only if there exists a strict ordering $\prec$ on $N$ such that
(i) for all $i, j \in N$, if $v(N) - v(N \setminus \{i\}) < v(N) - v(N \setminus \{j\})$, then $i \prec j$;
(ii) for all $i \in N$, if $\big\{j \in N \bigm| j \prec i\big\} \neq \emptyset$, then

$$x_i \leq v\left(\big\{j \in N \bigm| j \prec i\big\} \cup \big\{i\big\}\right) - v\left(\big\{j \in N \bigm| j \prec i\big\}\right).$$

Again, since the marginal contribution vectors are in the core on this domain, $\varphi^{**}$ satisfies *non-emptiness*. Note that it coincides with the core when $|N| \leq 2$. Figure 1 illustrates a case in which $\varphi^{**}(v)$ does not coincides with the core, and there are two strict orderings that satisfy condition (i) above. This solution trivially satisfies *anonymity* and *individual rationality*. We show that it also satisfies *complement consistency*.

**Lemma 3.** On the domain of convex games, $\varphi^{**}$ satisfies *complement consistency*.

*Proof.* Let $N, N' \in \mathcal{N}$ with $N' \subset N$, $v \in \mathcal{V}^N_{vex}$, and $x \in \varphi^{**}(v)$. By the definition of $\varphi^{**}(v)$, there exists a strict ordering $\prec$ on $N$ such that
(i) for all $i, j \in N$, if $v(N) - v(N \setminus \{i\}) < v(N) - v(N \setminus \{j\})$, then $i \prec j$;
(ii) for all $i \in N$, if $\big\{j \in N \bigm| j \prec i\big\} \neq \emptyset$, then

$$x_i \leq v\left(\big\{j \in N \bigm| j \prec i\big\} \cup \big\{i\big\}\right) - v\left(\big\{j \in N \bigm| j \prec i\big\}\right).$$

Since $x \in C(v)$ and the core is *complement consistent*, we have $r^x_{N'}(v) \in \mathcal{V}^{N'}_{vex}$ and $x_{N'} \in C(r^x_{N'}(v))$. We want to show that $x_{N'} \in \varphi^{**}(r^x_{N'}(v))$. If $|N'| \leq 2$, then $\varphi^{**}(r^x_{N'}(v)) = C(r^x_{N'}(v))$, and we are done.

Suppose that $|N'| \geq 3$. Note that for all $i \in N'$, since $N' \setminus \{i\} \neq \emptyset$, we have

$$\begin{aligned} & r^x_{N'}(v)(N') - r^x_{N'}(v)(N' \setminus \{i\}) \\ = \;& v(N) - \sum_{j \in N \setminus N'} x_j - v((N' \setminus \{i\}) \cup (N \setminus N')) + \sum_{j \in N \setminus N'} x_j \\ = \;& v(N) - v(N \setminus \{i\}). \end{aligned}$$

Thus, if $i, j \in N'$ are such that

$$r^x_{N'}(v)(N') - r^x_{N'}(v)(N' \setminus \{i\}) < r^x_{N'}(v)(N') - r^x_{N'}(v)(N' \setminus \{j\}),$$

then $i \prec j$. Let $i \in N'$ and $S \equiv \left\{ j \in N \;\middle|\; j \prec i \right\}$. Since $v$ is convex,

$$v((S \cup \{i\}) \cup (N \setminus N')) - v(S \cup (N \setminus N')) \geq v(S \cup \{i\}) - v(S).$$

If $S \cap N' \neq \emptyset$, then

$$\begin{aligned} & r^x_{N'}(v)((S \cup \{i\}) \cap N') - r^x_{N'}(v)(S \cap N') \\ = \;& v((S \cup \{i\}) \cup (N \setminus N')) - \sum_{j \in N \setminus N'} x_j - v(S \cup (N \setminus N')) + \sum_{j \in N \setminus N'} x_j \\ = \;& v((S \cup \{i\}) \cup (N \setminus N')) - v(S \cup (N \setminus N')) \\ \geq \;& v(S \cup \{i\}) - v(S) \\ \geq \;& x_i. \end{aligned}$$

Thus, $x_{N'} \in \varphi^{**}(r^x_{N'}(v))$. □

So, we have the following answer to Question 2:

**Proposition 2.** On the domain of convex games, the core is **not** the only solution that satisfies *non-emptiness*, *individual rationality*, *complement consistency*, and *anonymity*.

Given Propositions 1 and 2, one may wonder what axioms could be added to either of these lists to obtain the core as the unique solution.[5] Here we

[5]We thank an associate editor for prompting us to work on this question.

provide a partial answer: by adding the following three axioms to the list of axioms that appear in Proposition 1, we can single out the core.

**Homogeneity**: For all $v, w \in \mathcal{V}^N$, all $a \in \mathbb{R}_{++}$, and all $x \in \varphi(v)$, if $w = \alpha v$, then $\alpha x \in \varphi(w)$.

**Closedness**: For all $v \in \mathcal{V}^N$, $\varphi(v)$ is a closed set.

**Converse max consistency** (Peleg, 1986): For all $N \in \mathcal{N}$ with $|N| \geq 3$, all $v \in \mathcal{V}^N$, and all $x \in \varphi(v)$, if for all $N' \subset N$ with $|N'| = 2$, we have $\hat{r}^x_{N'}(v) \in \mathcal{V}^{N'}$ and $x_{N'} \in \varphi(\hat{r}^x_{N'}(v))$, then $x \in \varphi(v)$.

**Proposition 3.** On the domain of convex games, the core is the only solution that satisfies *non-emptiness*, *individual rationality*, *super-additivity*, *anonymity*, *homogeneity*, *closedness*, *max consistency*, and *converse max consistency.*

We don't know whether *homogeneity* is independent from others. In this sense, this axiomatization is not complete. However, since *homogeneity* itself is a desirable and innocuous property, one can properly say that Proposition 3 essentially describes the implications of all other axioms.

Finally, although we have shown that two well-known axiomatizations break down if the domain is restricted to the class of convex games, we should mention that there is another axiomatization of the core on the domain of all TU games provided by Peleg (1986), which remains valid even on the domain of convex games.[6] It says that on this domain, the core is the only solution that satisfies *max consistency*, *converse max consistency*, and the additional axiom of "unanimity", which says that the solution should coincide with the core in the two-person case.

[6]We thank an anonymous referee for pointing out this fact.

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# Appendix

In this appendix, first, we prove Proposition 3, and then prove the claim that $\varphi^*$ satisfies *converse max consistency*. Consider the following axioms:

**Zero-independence**: For all $v, w \in \mathcal{V}^N$, all $b \in \mathbb{R}^N$ and all $x \in \varphi(v)$, if for all $S \in 2^N$, $w(S) = v(S) + \sum_{i \in S} b_i$, then $x + b \in \varphi(w)$.

**Convex-valuedness**: For all $v \in \mathcal{V}^N$, $\varphi(v)$ is a convex set.

**Lemma 4.** On the domain of convex games, if a solution satisfies *individual rationality* and *super-additivity*, then it satisfies *zero-independence*.

**Proof.** Let $\varphi$ be a solution on $\mathcal{V}_{vex}$ that satisfies *individual rationality* and *super-additivity*. Let $b \in \mathbb{R}^N$ and $v, w \in \mathcal{V}^N_{vex}$ be such that for all $S \in 2^N$, $w(S) = v(S) + \sum_{i \in S} b_i$. Let $x \in \varphi(v)$. We want to show $x + b \in \varphi(w)$.

For all $S \in 2^N$, let $w'(S) \equiv \sum_{i \in S} b_i$. Then $w' \in \mathcal{V}^N_{vex}$ and $w = v + w'$. Since $w'$ is additive and $\varphi$ is *individually rational*, $\varphi(w') = \{b\}$. By *super-additivity*, $x + b \in \varphi(v + w') = \varphi(w)$. □

**Lemma 5.** On the domain of convex games, if a solution satisfies *homogeneity* and *super-additivity*, then it is *convex-valued*.

**Proof.** Let $\varphi$ be a solution on $\mathcal{V}_{vex}$ that satisfies *homogeneity* and *super-additivity*. Let $N \in \mathcal{N}$, $v \in \mathcal{V}^N_{vex}$, $x, y \in \varphi(v)$, and $\lambda \in (0,1)$. We want to show that $\lambda x + (1-\lambda)y \in \varphi(v)$. Note that $\lambda v, (1-\lambda)v \in \mathcal{V}^N_{vex}$ and $v = \lambda v + (1-\lambda)v$. By *homogeneity*, $\lambda x \in \varphi(\lambda v)$ and $(1-\lambda)y \in \varphi((1-\lambda)v)$. By *super-additivity*, $\lambda x + (1-\lambda)y \in \varphi(\lambda v + (1-\lambda)v) = \varphi(v)$. □

The following lemma is due to Peleg (1986).

**Lemma 6.** On the domain of balanced games, if a solution satisfies *individual rationality* and *max consistency*, then it satisfies *efficiency*.

A similar claim for convex games can be proved in exactly the same way as Peleg's proof of Lemma 6. So, its proof is omitted.

**Lemma 7.** On the domain of convex games, if a solution satisfies *individual rationality* and *max consistency*, then it satisfies *efficiency*.

**Proof of Proposition 3.** The core satisfies the seven axioms. Let $\varphi$ be a solution on $\mathcal{V}_{vex}$ that satisfies the seven axioms. We show that $\varphi$ coincides with the core. Since both the core and $\varphi$ satisfy *converse max consistency*, it is enough to show that $\varphi$ coincides with the core in the two-person case.

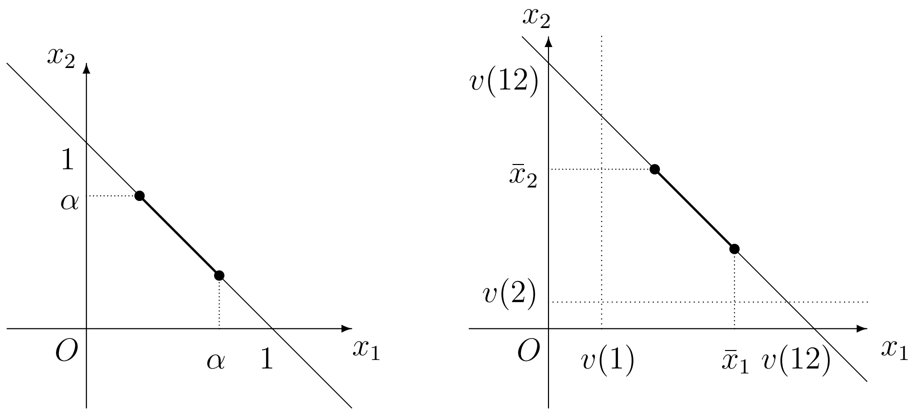


Figure 2: In (a), $v(1) = v(2) = 0$, $v(12) = 1$, and $\varphi(v) = \{x \in \mathbb{R}^{\{1,2\}} \mid 0 \leq x_i \leq \alpha\}$. In (b), $\bar{x}_1 = \alpha(v(12) - v(2)) + (1 - \alpha)v(1)$, $\bar{x}_2 = \alpha(v(12) - v(1)) + (1 - \alpha)v(2)$, and $\varphi(v) = \{x \in \mathbb{R}^{\{1,2\}} \mid v(i) \leq x_i \leq \bar{x}_i\}$.

By Lemmas 4, 5, and 7, $\varphi$ is *zero-independent*, *convex-valued*, and *efficient*. Let $N = \{1, 2\}$ and $v_0 \in \mathcal{V}^N_{vex}$ be such that $v_0(1) = v_0(2) = 0$ and $v_0(12) = 1$. Let $\alpha \equiv \max_{(x_1,x_2)\in\varphi(v_0)} x_1$. By *efficiency* and *individual rationality*, $\varphi(v_0)$ is bounded. By *closedness*, $\varphi(v_0)$ is a closed set. Thus, $\alpha$ is well-defined. By *anonymity*, $(1-\alpha, \alpha) \in \varphi(v_0)$. By *convex-valuedness*, $\varphi(v_0)$ is the interval connecting $(\alpha, 1 - \alpha)$ and $(1 - \alpha, \alpha)$. By *non-emptiness*, $\alpha \in \left[\frac{1}{2}, 1\right]$. By *zero-independence*, *homogeneity*, and *anonymity*, for all $N \in \mathcal{N}$ with $|N| = 2$ and all $v \in \mathcal{V}^N_{vex}$,

$$\varphi(v) = \left\{ x \in C(v) \;\middle|\; \text{for all } i \in N,\ x_i - v(\{i\}) \leq \alpha\left[v(N) - \sum_{j\in N} v(\{j\})\right] \right\}.$$

First, suppose that $\alpha = \frac{1}{2}$. Then $\varphi$ coincides with the "standard solution" in the two-person case: for all $N \in \mathcal{N}$ with $|N| = 2$, all $v \in \mathcal{V}^N_{vex}$, and all $i \in N$,

$$\varphi_i(v) = v(\{i\}) + \frac{1}{2}\left[v(N) - \sum_{j\in N} v(\{j\})\right].$$

On the domain of convex games, the nucleolus (Schmeidler, 1969) is the only solution that coincides with the standard solution in the two-person case and satisfies *max consistency.*[7] Thus, $\varphi$ coincides with the nucleolus. However, the nucleolus violates *super-additivity*. Thus, $\alpha$ cannot be $\frac{1}{2}$.

Next, suppose that $\frac{1}{2} < \alpha \leq 1$. Then, let $N \equiv \{1,2,3\}$ and $v \in \mathcal{V}^N_{vex}$ be such that $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2,3\}) = 0$, $v(\{1,2\}) = v(\{1,3\}) = 1$, and $v(N) = 3$. We show that if $\frac{1}{2} < \alpha < 1$, then $\varphi(v)$ is not convex.

Let

$$\begin{aligned} x^\alpha &\equiv \left(\frac{3\alpha}{1+\alpha}, \frac{3\alpha}{1+\alpha}, \frac{3(1-\alpha)}{1+\alpha}\right), \\ y^\alpha &\equiv (4\alpha - 1, 2(1-\alpha), 2(1-\alpha)). \end{aligned}$$

First, we show that $(x^\alpha_1, x^\alpha_2) \in \varphi(r^{x^\alpha}_{\{1,2\}}(v))$. Note that

$$\begin{aligned} \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,3\}) - x^\alpha_3\} \\ &= \max\left\{0, 1 - \frac{3(1-\alpha)}{1+\alpha}\right\} \\ &= \max\left\{0, \frac{2(2\alpha-1)}{1+\alpha}\right\} \\ &= \frac{2(2\alpha-1)}{1+\alpha}, \\ \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{2,3\}) - x^\alpha_3\} \\ &= \max\left\{0, -\frac{3(1-\alpha)}{1+\alpha}\right\} \\ &= 0, \\ \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1,2\}) &= v(\{1,2,3\}) - x^\alpha_3 \\ &= 3 - \frac{3(1-\alpha)}{1+\alpha} \\ &= \frac{6\alpha}{1+\alpha}. \end{aligned}$$

[7]This result follows from the fact that the nucleolus coincides with the prekernel on this domain (Maschler, Peleg, and Shapley, 1972), and the prekernel satisfies *converse max consistency*. See Hokari (2005) for more detail.

Thus,

$$\begin{aligned}
& x_1^\alpha - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1,2\}) - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1\}) - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{2\})\right] \\
&= \frac{3\alpha}{1+\alpha} - \frac{2(2\alpha-1)}{1+\alpha} - \alpha\left[\frac{6\alpha}{1+\alpha} - \frac{2(2\alpha-1)}{1+\alpha}\right] \\
&= -\frac{(\alpha+2)(2\alpha-1)}{1+\alpha} \\
&< 0, \\
& x_2^\alpha - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{2\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1,2\}) - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{1\}) - \hat{r}^{x^\alpha}_{\{1,2\}}(v)(\{2\})\right] \\
&= \frac{3\alpha}{1+\alpha} - \alpha\left[\frac{6\alpha}{1+\alpha} - \frac{2(2\alpha-1)}{1+\alpha}\right] \\
&= \frac{\alpha(1-2\alpha)}{1+\alpha} \\
&< 0.
\end{aligned}$$

Hence, $(x_1^\alpha, x_2^\alpha) \in \varphi(\hat{r}^{x^\alpha}_{\{1,2\}}(v))$.

Next, we show that $(x_1^\alpha, x_3^\alpha) \in \varphi(\hat{r}^{x^\alpha}_{\{1,3\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,2\}) - x_2^\alpha\} \\
&= \max\left\{0, 1 - \frac{3\alpha}{1+\alpha}\right\} \\
&= \max\left\{0, \frac{1-2\alpha}{1+\alpha}\right\} \\
&= 0, \\
\hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{3\}) &= \max\{v(\{3\}), v(\{2,3\}) - x_2^\alpha\} \\
&= \max\left\{0, -\frac{3\alpha}{1+\alpha}\right\} \\
&= 0, \\
\hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1,3\}) &= v(\{1,2,3\}) - x_2^\alpha \\
&= 3 - \frac{3\alpha}{1+\alpha} \\
&= \frac{3}{1+\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&x_1^\alpha - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{3\})\right] \\
&= \frac{3\alpha}{1+\alpha} - \alpha\cdot\frac{3}{1+\alpha} \\
&= 0, \\
&x_3^\alpha - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{3\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{x^\alpha}_{\{1,3\}}(v)(\{3\})\right] \\
&= \frac{3(1-\alpha)}{1+\alpha} - \alpha\cdot\frac{3}{1+\alpha} \\
&= \frac{3(1-2\alpha)}{1+\alpha} \\
&< 0.
\end{aligned}$$

Hence, $(x_1^\alpha, x_3^\alpha) \in \varphi(\hat{r}^{x^\alpha}_{\{1,3\}}(v))$.

Next, we show that $(x_2^\alpha, x_3^\alpha) \in \varphi(\hat{r}^{x^\alpha}_{\{2,3\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{1,2\}) - x_1^\alpha\} \\
&= \max\left\{0, 1 - \frac{3\alpha}{1+\alpha}\right\} \\
&= \max\left\{0, \frac{1-2\alpha}{1+\alpha}\right\} \\
&= 0, \\
\hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{3\}) &= \max\{v(\{3\}), v(\{1,3\}) - x_1^\alpha\} \\
&= \max\left\{0, 1 - \frac{3\alpha}{1+\alpha}\right\} \\
&= 0, \\
\hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2,3\}) &= v(\{1,2,3\}) - x_1^\alpha \\
&= 3 - \frac{3\alpha}{1+\alpha} \\
&= \frac{3}{1+\alpha}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& x_2^\alpha - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2,3\}) - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2\}) - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{3\})\right] \\
= & \frac{3\alpha}{1+\alpha} - \alpha \cdot \frac{3}{1+\alpha} \\
= & 0, \\
& x_3^\alpha - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{3\}) - \alpha\left[\hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2,3\}) - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{2\}) - \hat{r}^{x^\alpha}_{\{2,3\}}(v)(\{3\})\right] \\
= & \frac{3(1-\alpha)}{1+\alpha} - \alpha \cdot \frac{3}{1+\alpha} \\
= & \frac{3(1-2\alpha)}{1+\alpha} \\
< & 0.
\end{aligned}$$

Hence, $(x_2^\alpha, x_3^\alpha) \in \varphi(\hat{r}^{x^\alpha}_{\{2,3\}}(v))$.

By *converse max consistency*, $x^\alpha \in \varphi(v)$.

Next, we show that $(y_1^\alpha, y_2^\alpha) \in \varphi(r^{y^\alpha}_{\{1,2\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,3\}) - y_3^\alpha\} \\
&= \max\{0, 1 - 2(1-\alpha)\} \\
&= \max\{0, 2\alpha - 1\} \\
&= 2\alpha - 1, \\
\hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{2\}) &= \max\{v(\{2\}, v(\{2,3\}) - y_3^\alpha\} \\
&= \max\{0, -2(1-\alpha)\} \\
&= 0, \\
\hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1,2\}) &= v(\{1,2,3\}) - y_3^\alpha \\
&= 3 - 2(1-\alpha) \\
&= 2\alpha + 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
& y_1^\alpha - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1\}) - \alpha \left[\hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1,2\}) - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1\}) - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{2\})\right] \\
= \;& 4\alpha - 1 - 2\alpha + 1 - \alpha(2\alpha + 1 - 2\alpha + 1) \\
= \;& 0, \\
& y_2^\alpha - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{2\}) - \alpha \left[\hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1,2\}) - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{1\}) - \hat{r}^{y^\alpha}_{\{1,2\}}(v)(\{2\})\right] \\
= \;& 2(1-\alpha) - \alpha(2\alpha + 1 - 2\alpha + 1) \\
= \;& 2(1-2\alpha) \\
< \;& 0.
\end{aligned}$$

Hence, $(y_1^\alpha, y_2^\alpha) \in \varphi(\hat{r}^{y^\alpha}_{\{1,2\}}(v))$.

Next, we show that $(y_1^\alpha, y_3^\alpha) \in \varphi(\hat{r}^{y^\alpha}_{\{1,3\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,2\}) - y_2^\alpha\} \\
&= \max\{0, 1 - 2(1-\alpha)\} \\
&= \max\{0, 2\alpha - 1\} \\
&= 2\alpha - 1, \\
\hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{3\}) &= \max\{v(\{3\}, v(\{2,3\}) - y_2^\alpha\} \\
&= \max\{0, -2(1-\alpha)\} \\
&= 0, \\
\hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1,3\}) &= v(\{1,2,3\}) - y_2^\alpha \\
&= 3 - 2(1-\alpha) \\
&= 2\alpha + 1.
\end{aligned}$$

Thus,

$$\begin{aligned}
& y_1^\alpha - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1\}) - \alpha \left[\hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{3\})\right] \\
= \;& 4\alpha - 1 - 2\alpha + 1 - \alpha(2\alpha + 1 - 2\alpha + 1) \\
= \;& 0, \\
& y_3^\alpha - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{3\}) - \alpha \left[\hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{y^\alpha}_{\{1,3\}}(v)(\{3\})\right] \\
= \;& 2(1-\alpha) - \alpha(2\alpha + 1 - 2\alpha + 1) \\
= \;& 2(1-2\alpha) \\
< \;& 0.
\end{aligned}$$

Hence, $(y_1^\alpha, y_3^\alpha) \in \varphi(\hat{r}^{y^\alpha}_{\{1,3\}}(v))$.

Next, we show that $(y_2^\alpha, y_3^\alpha) \in \varphi(\hat{r}^{y^\alpha}_{\{2,3\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2\}) &= \max\{v(\{2\}), v(\{1,2\}) - y_1^\alpha\} \\
&= \max\{0, 1 - 4\alpha + 1\} \\
&= \max\{0, 2(1 - 2\alpha)\} \\
&= 0, \\
\hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{3\}) &= \max\{v(\{3\}, v(\{1,3\}) - y_1^\alpha\} \\
&= \max\{0, 1 - 4\alpha + 1\} \\
&= 0, \\
\hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2,3\}) &= v(\{1,2,3\}) - y_1^\alpha \\
&= 3 - 4\alpha + 1 \\
&= 4(1 - \alpha).
\end{aligned}$$

Thus,

$$\begin{aligned}
& y_2^\alpha - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2\}) - \alpha\left[\hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2,3\}) - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2\}) - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{3\})\right] \\
&= 2(1 - \alpha) - \alpha \cdot 4(1 - \alpha) \\
&= 2(\alpha - 1)(2\alpha - 1) \\
&\leq 0, \\
& y_3^\alpha - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{3\}) - \alpha\left[\hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2,3\}) - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{2\}) - \hat{r}^{y^\alpha}_{\{2,3\}}(v)(\{3\})\right] \\
&= 2(1 - \alpha) - \alpha \cdot 4(1 - \alpha) \\
&= 2(\alpha - 1)(2\alpha - 1) \\
&\leq 0.
\end{aligned}$$

Hence, $(y_2^\alpha, y_3^\alpha) \in \varphi(\hat{r}^{y^\alpha}_{\{2,3\}}(v))$.

By *converse max consistency*, $y^\alpha \in \varphi(v)$.

Let

$$z^\alpha \equiv \frac{x^\alpha + y^\alpha}{2}.$$

We show that if $\frac{1}{2} < \alpha < 1$, then $(z_1^\alpha, z_3^\alpha) \notin \varphi(\hat{r}^{z^\alpha}_{\{1,3\}}(v))$. Note that

$$\begin{aligned}
\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) &= \max\{v(\{1\}), v(\{1,2\}) - z_2^\alpha\} \\
&= \max\left\{0, 1 - \frac{3\alpha}{2(1+\alpha)} - 1 + \alpha\right\} \\
&= \max\left\{0, \frac{\alpha(2\alpha-1)}{2(1+\alpha)}\right\} \\
&= \frac{\alpha(2\alpha-1)}{2(1+\alpha)}, \\
\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{3\}) &= \max\{v(\{3\}), v(\{2,3\}) - z_2^\alpha\} \\
&= \max\left\{0, 0 - \frac{3\alpha}{2(1+\alpha)} - 1 + \alpha\right\} \\
&= \max\left\{0, -\frac{3\alpha}{2(1+\alpha)} - (1-\alpha)\right\} \\
&= 0, \\
\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1,3\}) &= v(\{1,2,3\}) - z_2^\alpha \\
&= 3 - \frac{3\alpha}{2(1+\alpha)} - 1 + \alpha \\
&= \frac{2\alpha^2 + 3\alpha + 4}{2(1+\alpha)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&z_1^\alpha - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) - \alpha\left[\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{3\})\right] \\
&= z_1^\alpha - (1-\alpha)\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) - \alpha\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1,3\}) + \alpha\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{3\}) \\
&= \frac{3\alpha}{2(1+\alpha)} + 2\alpha - \frac{1}{2} - (1-\alpha)\cdot\frac{\alpha(2\alpha-1)}{2(1+\alpha)} - \alpha\cdot\frac{2\alpha^2+3\alpha+4}{2(1+\alpha)} \\
&= -\frac{(\alpha-1)(2\alpha-1)}{2(1+\alpha)}.
\end{aligned}$$

Now, suppose that $\frac{1}{2} < \alpha < 1$. Then

$$z_1^\alpha - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) - \alpha\left[\hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1,3\}) - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{1\}) - \hat{r}^{z^\alpha}_{\{1,3\}}(v)(\{3\})\right] \;>\; 0.$$

This implies $(z_1^\alpha, z_3^\alpha) \notin \varphi(\hat{r}^{z^\alpha}_{\{1,3\}}(v))$, and hence, $z^\alpha \notin \varphi(v)$, which contradicts *convex-valuedness* of $\varphi$. (The arguments above are illustrated in Figures 3 and 4 for the case of $\alpha = \frac{3}{4}$.)

So, we conclude that $\alpha = 1$, which implies that $\varphi$ coincides with the core in the two-person case.

Let $N \in \mathcal{N}$ with $|N| \geq 3$ and $v \in \mathcal{V}_{vex}^N$. We show that $\varphi(v) = C(v)$. Let $x \in \varphi(v)$. Since $\varphi$ is *max consistent* and $\varphi$ coincides with the core in the two-person case, for all $N' \subset N$ with $|N'| = 2$, we have $\hat{r}_{N'}^x(v) \in \mathcal{V}_{vex}^{N'}$ and $x_{N'} \in \varphi(\hat{r}_{N'}^x(v)) = C(\hat{r}_{N'}^x(v))$. Since the core is *conversely max consistent*, we have $x \in C(v)$. Thus, $\varphi(v) \subseteq C(v)$.

Next, let $y \in C(v)$. Since the core is *max consistent* and the core coincides with $\varphi$ in the two-person case, for all $N' \subset N$ with $|N'| = 2$, we have $\hat{r}_{N'}^y(v) \in \mathcal{V}_{vex}^{N'}$ and $y_{N'} \in C(\hat{r}_{N'}^y(v)) = \varphi(\hat{r}_{N'}^y(v))$. Since $\varphi$ is *conversely max consistent*, we have $x \in \varphi(v)$. Thus, $C(v) \subseteq \varphi(v)$.

So, we conclude that $\varphi$ coincides with the core on the whole domain. $\square$

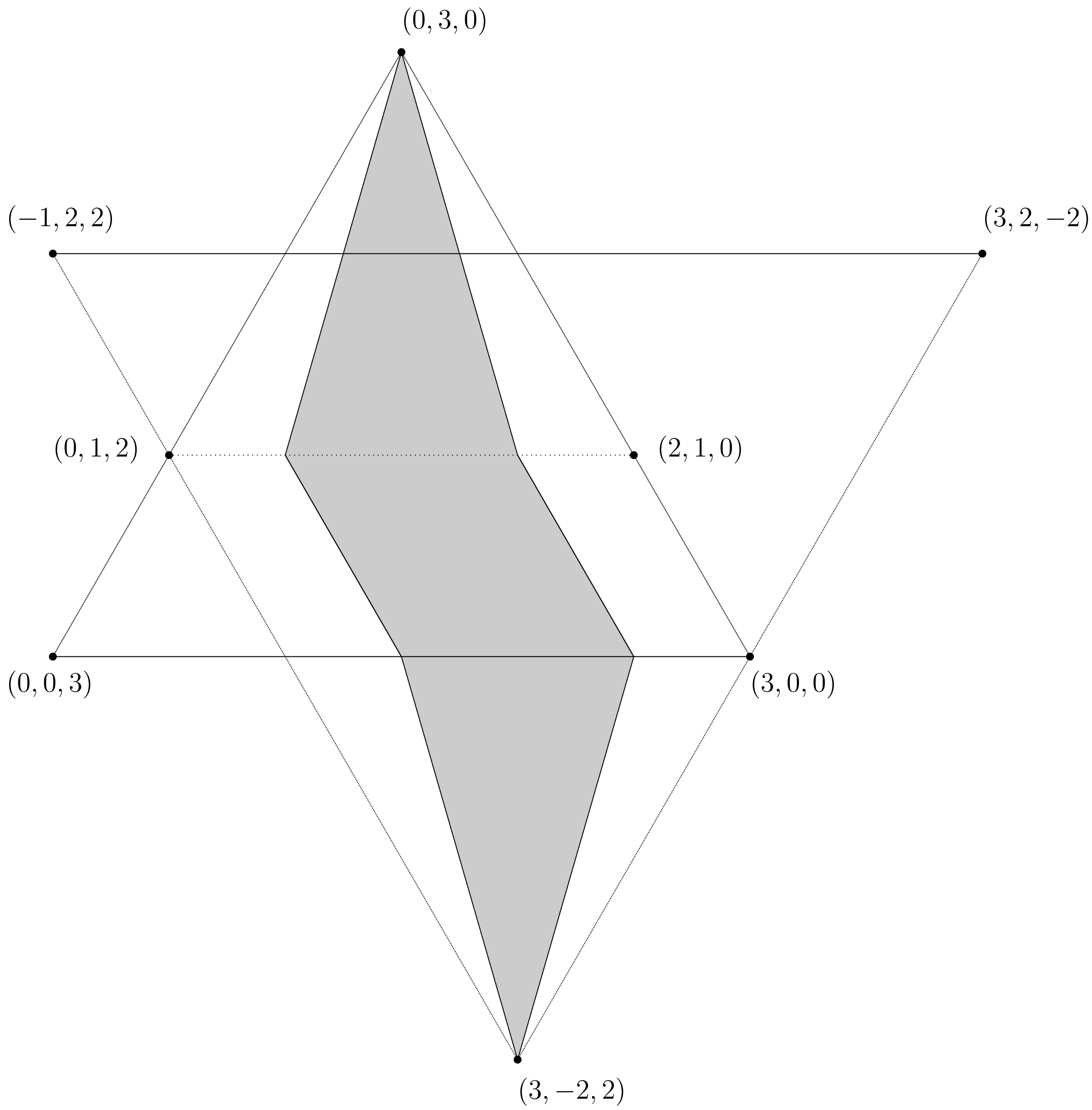


Figure 3: $\alpha = \frac{3}{4}$. The shaded area describes the set of payoff vectors such that $(x_1, x_3) \in \varphi\left(\hat{r}^x_{\{1,3\}}(v)\right)$.

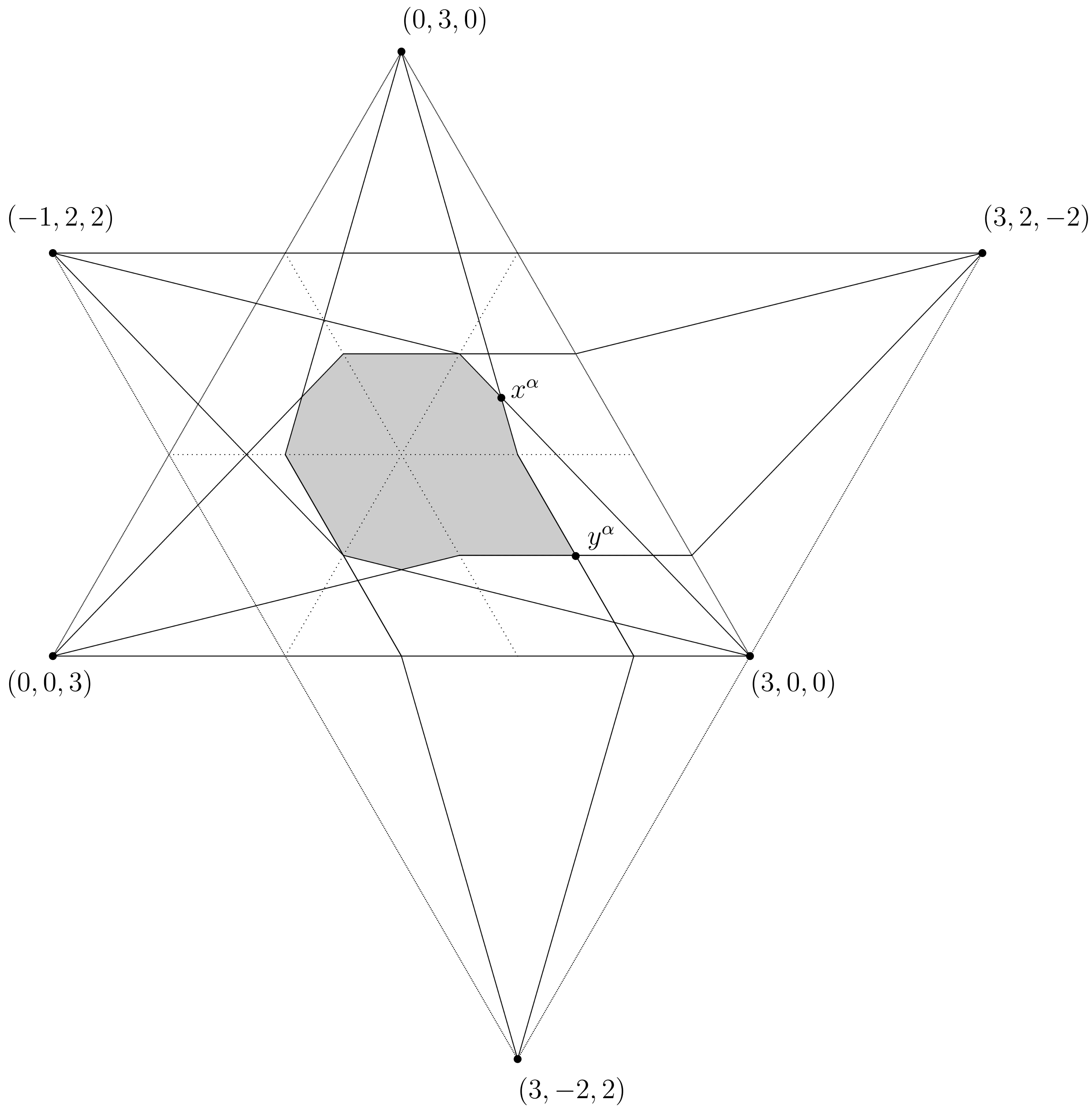


Figure 4: $\alpha = \frac{3}{4}$. The shaded area describes the set of payoff vectors such that $(x_1, x_2) \in \varphi(\hat{r}^x_{\{1,2\}}(v))$, $(x_1, x_3) \in \varphi(\hat{r}^x_{\{1,3\}}(v))$, and $(x_2, x_3) \in \varphi(\hat{r}^x_{\{2,3\}}(v))$.

Next, we prove the following claim:

**Claim 1.** On the domain of convex games, $\varphi^*$ satisfies *converse max consistency.*

We use the following lemma in the proof.

**Lemma 8.** Let $v \in \mathcal{V}^N_{vex}$, $x \in C(v)$, and $S, T \in 2^N$. If $x(S) = v(S)$ and $x(T) = v(T)$, then $x(S \cap T) = v(S \cap T)$ and $x(S \cup T) = v(S \cup T)$.

**Proof.** Since $v$ is convex,

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T) = x(S) + x(T) = x(S \cup T) + x(S \cap T).$$

Since $x \in C(v)$, we have $x(S \cup T) \geq v(S \cup T)$ and $x(S \cap T) \geq v(S \cap T)$. Thus, $x(S \cup T) = v(S \cup T)$ and $x(S \cap T) = v(S \cap T)$. □

**Proof of Claim 1.** Let $N \in \mathcal{N}$ with $|N| \geq 3$, $v \in \mathcal{V}^N_{vex}$, and $x \in \mathbb{R}^N$ be such that for all $N' \subset N$ with $|N'| = 2$, we have $\hat{r}^x_{N'}(v) \in \mathcal{V}^{N'}_{vex}$ and $x_{N'} \in \varphi^*(\hat{r}^x_{N'}(v))$.

Since $\varphi^*$ is a subsolution of the core and the core is *conversely max consistent*, $x \in C(v)$. Suppose that there exists $S \notin \mathcal{S}(v)$ such that $S \neq N$ and $x(S) = v(S)$. Let $i \in S$. Note that for all $j \in N \setminus S$,

$$\hat{r}^x_{\{i,j\}}(v)(\{i\}) = \max_{T \subseteq N \setminus \{i,j\}} [v(\{i,j\} \cup T) - x(T)] \geq v(S) - x(S \setminus \{i\}) = x_i.$$

Since $(x_i, x_j) \in \varphi^*(\hat{r}^x_{\{i,j\}}(v))$ and $\varphi^*$ is *individually rational*, $x_i = \hat{r}^x_{\{i,j\}}(v)(\{i\})$. This implies $x_j = \hat{r}^x_{\{i,j\}}(v)(\{j\})$. Thus there exists $T_{ij} \subset N$ such that $j \in T_{ij}$, $i \notin T_{ij}$, and $x(T_{ij}) = v(T_{ij})$. Let $T_i \equiv \bigcup_{j \in N \setminus S} T_{ij}$. Then, by Lemma 8, $x(T_i) = v(T_i)$.

Note that $N \setminus S = \bigcap_{i \in S} T_i$. Again by Lemma 8, $x(N \setminus S) = v(N \setminus S)$. This implies

$$v(S) + v(N \setminus S) = x(S) + x(N \setminus S) = x(N) = v(N).$$

Thus, for all $y \in C(v)$, $y(S) = v(S)$, which contradicts $S \notin \mathcal{S}(v)$. □