Relationship between the Shapley value and other solutions

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Abstract

The main contribution of this paper is to give a necessary and sufficient condition under which the Shapley value coincides with the nucleolus, the CIS value and the ENSC value in the following classes of games; airport games (Littlechild and Owen (1973)), bidder collusion games (Graham et al. (1990)) and polluted river games (Ni and Wang (2007)). Along the way, we also show the necessary and sufficient condition under which the Shapley value coincides with the prenucleolus for general 3-person game. The general coincidence condition for the CIS value and the ENSC value is also given.

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1 Introduction

The Shapley value is one of the most well-known solution concepts of TU games. It is widely applied to several economic situations. For example, we can refer the following problems: a problem to determine the allocation of the cost of constructing a runway in an airport which is called an airport game introduced by Littlechild and Owen (1973); a problem of dividing the surplus

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of collusion in auction which is called a bidder-collusion game introduced by Graham et al. (1990); a problem to allocate the cost of polluting a river among several firms which is called a polluted river game introduced by Ni and Wang (2007). One of the special features of the Shapley value is that it always determines one suitable allocation which satisfies several good properties. On the other hand, there are other several one-point solutions in TU games, the nucleolus, the CIS value and the ENSC value. This paper is devoted to the investigation of necessary and sufficient condition under which the Shapley value coincides with another solution in the class of games mentioned above.

The interesting feature of approaching to the economic situations from cooperative game theory is that the theoretical results are closely related to the results in the real situations. For example, Littlechild and Thompson (1977) applied the theory to the airport at Barmingham in England and showed that theoretic solutions such as the Shapley value and the nucleolus propose a similar cost allocation to the real data. Ni and Wang (2007) started their discussion of polluted river from the doctrines in international disputes and finally obtained the formula of the Shapley value.

When we discuss the relationship between the real situations and theoretical results, there is one disputing problem; which solution concept should we apply? For example, consider the situation in which we try to insist that the Shapley value has a close relationship with the real situations. In this case, however, it may be the case that the real situation is more closely related to other solutions, such as the nucleolus, the CIS value or the ENSC value. In this case, to discuss the relationship from the Shapley value is not suitable. One way to overcome the problem is to show that the Shapley value and another solution prescribe the same outcome. If we can show this statement, then we do not need to care about which solution is more suitable. As a result, the precise characterization of the coincidence condition in the class of games is a critical task.

There are two other benefits of analyzing the coincidence condition among solutions. The first benefit is that we can check the desirability of the Shapley value in a broader sense. Under the coincidence condition, the outcome obtained by the Shapley value can also be obtained from different solutions, and the outcome seems desirable. On the other hand, if the condition is violated, then we can know that the Shapley value is not desirable from the viewpoint of other solutions. If we try to apply the Shapley value to some economic situations, knowing the desirability of the Shapley value is essential. The second benefit is that we can find simpler way of calculating solutions. For example, if the Shapley value and the nucleolus coincide, then the lexicographic minimization problem can be solved by calculating the

Shapley value. Similarly, if the Shapley value coincides with the CIS value or the ENSC value, then the Shapley value can be calculated in a simpler way.

The sufficient condition of coincidence between the Shapley value and the (pre)nucleolus has been investigated by the following papers; Chun and Hokari (2007), Kar et al. (2009), Chang and Tseng (2011). The main contribution of this paper is that we can give not only sufficient, but also necessary conditions. The way of proving the coincidence condition in this paper is also new. Although the previous researches discuss the coincidence with fixing the set of players, we inductively increase the number of players. We first focus on the coincidence condition for 3-person games. We show that the set of all 0-normalized 3-person games where the Shapley value and the prenucleolus coincide is equivalent to the union of the set of all symmetric games and the set of all games satisfying PS property, which was introduced by Kar, Mitra and Mutuswami (2009). We also show that the set of all 0-normalized 3-person games where the Shapley value and the CIS value, the ENSC value coincide is equivalent to the set of all symmetric games. We generalize the result for 3-person games to general n-person games by using reduced game property.

The proof of the coincidence for 3-person games highly relies on the new basis introduced by Yokote, Funaki and Kamijo (2013). The brief explanation of the new basis is given as well. The benefit of using the basis is that we can express the excess game with respect to the Shapley value only by using simple games. This simplicity is useful when we prove the coincidence condition.

This paper is organized as follows. Section 2 contains notations and definitions. In Section 3, we discuss the necessary and sufficient condition under which the Shapley value coincides with another solution which satisfies Weak Strategic Invariance in 3-person games. In Section 4, we derive the coincidence condition in some specific classes and discuss the result. Section 5 gives concluding remarks.

2 Notations and definitions

2.1 Game and coalition

For any two sets A and B, $A \subset B$ means that A is a proper subset of B. $A \subseteq B$ means that $A \subset B$ or $A = B$. Let $|A|$ denote the cardinality of A. Let $N \subset \mathbb{N}$ be a finite set of players, and we call $S \subseteq N$ as a coalition of N. We define $|N| = n$, and we restrict our attention to games with no less than

2 players. A characteristic function $v : 2^N \to \mathbb{R}$ assigns a real number to each coalition of N, and satisfies $v(\emptyset) = 0$. We call $v(S)$ as the worth of coalition S. The pair (N, v) is called a game, and the set of all games with player set N is denoted as Γ^N. We write v instead of (N, v) if the set of players is clear. For any game (N, v), let (S, v) denote the restriction of v on S.

For any v, $w \in \Gamma^N$ and $\alpha \in \mathbb{R}$, we define the sum and the scalar multiplication of games $v + w \in \Gamma^N$, αv, as follows: $(v + w)(S) = v(S) + w(S)$ for all $S \subseteq N, S \neq \emptyset$, $(\alpha v)(S) = \alpha v(S)$ for all $S \subseteq N$, $S \neq \emptyset$. Then, we can identify Γ^N as $\mathbb{R}^{2^n-1}$.

A game $v \in \Gamma^N$ is called simple if $v(S) = 0$ or 1 for all $S \subseteq N$. A game $v \in \Gamma^N$ is called convex if $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$ for all S, $T \subseteq N$. A game is called symmetric if the worth of each coalition depends only on the number of players in the coalition. For any game $v \in \Gamma^N$ and $\beta \in \mathbb{R}^n$, the game $v + \beta \in \Gamma^N$ is given by $(v + \beta)(S) = v(S) + \sum_{i \in S} \beta_i$ for all $S \subseteq N$, $S \neq \emptyset$. For any game $v \in \Gamma^N$, let v^0 denote the game $v - \beta$, where $\beta \in \mathbb{R}^n$, $\beta_i = v(\{i\})$ for all $i \in N$. The game v^0 is called the 0-normalized game of v. For any $v \in \Gamma^N$, let v^d denote the dual game of v, which is given by $v^d(S) = v(N) - v(N \backslash S)$ for all $S \subseteq N$, $S \neq \emptyset$. Let $e_S \in \Gamma^N$, $S \subseteq N$, $S \neq \emptyset$, denote the following game: $e_S(S) = 1$ and $e_S(T) = 0$ for all $T \subseteq N$, $T \neq S$, $T \neq \emptyset$.

Take any $v \in \Gamma^N$ and $x \in \mathbb{R}^n$. Suppose that each coordinate i, $i = 1, \ldots, n$, of x corresponds to the amount which player i receives in x. We define the excess of coalition $S \subseteq N$, $S \neq \emptyset$, with respect to x in the game v as follows: $e(S, x, v) = v(S) - \sum_{i \in S} x_i$.

2.2 Solutions and axioms

Let $PI(v)$ denote the preimputation set of $v \in \Gamma^N$, which is given by

$$PI(v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N)\}.$$

A solution prescribes an element of $PI(v)$ to each game $v \in \Gamma^N$. We define five solutions. The Shapley value, introduced by Shapley (1953), is defined as follows: for any $v \in \Gamma^N$,

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{(n - |S|)!(|S| - 1)!}{n!} \big(v(S) - v(S \backslash \{i\})\big),$$

for all $i \in N$. The Shapley value ϕ is a linear function from $\mathbb{R}^{2^n-1}$ to $\mathbb{R}^n$. We define the dividend which was introduced by Harsanyi (1959). For any

$v \in \Gamma^N$,

$$D(S,v) = \sum_{T \subseteq S} (-1)^{|S|-|T|} v(T),$$

for all $S \subseteq N, S \neq \emptyset$. The following equation holds: for any $v \in \Gamma^N$,

$$\phi_i(v) = \sum_{S \subseteq N : i \in S} \frac{1}{|S|} D(S,v),$$

for all $i \in N$. Behind the payoff vector given by the Shapley value, we can consider the situation in which each player enters a room in an order and receives the marginal contribution. If we assume that every order occurs with the equal probability, then the expected value is equal to the Shapley value.

For any $x, y \in \mathbb{R}^n$, $y \geq_{lex} x$ means that y is greater than x in the lexicographic ordering of $\mathbb{R}^n$. Let $\theta(x) = (\theta_1(x), \theta_2(x), \ldots, \theta_{2^n-2}(x)) \in \mathbb{R}^{2^n-2}$ denote the sequence of excess of $S \subset N$, $S \neq \emptyset$, with respect to x, where $\theta_t(x) \geq \theta_{t+1}(x)$ for all $t, 1 \leq t \leq 2^n - 3$. The nucleolus Nu, introduced by Schmeidler (1969), is defined as follows: for any $v \in \Gamma^N$,

$$Nu(v) = \{x \in I(v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in I(v)\},$$

where

$$I(v) = \{x \in \mathbb{R}^n : \sum_{i \in N} x_i = v(N), x_i \geq v(\{i\}) \text{ for all } i \in N\}.$$

In the remaining part of this paper, we mainly consider convex games. In the class of games, the nucleolus coincides with the following solution, the prenucleolus.

$$PN(v) = \{x \in PI(v) : \theta(y) \geq_{lex} \theta(x) \text{ for all } y \in PI(v)\}.$$

The prenucleolus always prescribes a single element of the preimputation set. The desirability of the prenucleolus is that the solution lexicographically minimizes the excess of each coalition, and the outcome seems fair.

The next two solutions, the CIS value and the ENSC value, were introduced by Driessen and Funaki (1991). The CIS value is defined as follows: for any $v \in \Gamma^N$,

$$CIS_i(v) = v(\{i\}) + \frac{v(N) - \sum_{j \in N} v(\{j\})}{n},$$

for all $i \in N$. The CIS value is sometimes called as the equal surplus division rule. The ENSC value is defined as follows: for any $v \in \Gamma^N$,

$$ENSC_i(v) = v(N) - v(N \backslash \{i\}) + \frac{v(N) - \sum_{j \in N} \big(v(N) - v(N \backslash \{j\})\big)}{n},$$

for all $i \in N$. The above two solutions are defined based on the two steps below.

Step 1: The players receive the amount which they can guarantee by their own.

Step 2: The remainder is divided equally among the players.

In the case of the CIS value, we assume that each player i can guarantee $v(\{i\})$. In the case of the ENSC value, we assume that each player i can guarantee $v(N) - v(N \backslash \{i\})$.

When we prove the coincidence condition, reduced game property plays an important role. Let $v \in \Gamma^N$, $x \in PI(v)$ and $S \subset N$, $S \neq \emptyset$. We define three types of reduced game.[1] First, we define the M-reduced game $(S, v^{M,x})$ on S introduced by Davis and Maschler (1965).

$$v^{M,x}(T) = \begin{cases} v(N) - \sum_{i \in N \backslash S} x_i & \text{if } T = S, \\ \max\{v(T \cup R) - \sum_{i \in R} x_i : R \subseteq N \backslash S, R \neq \emptyset\} & \text{if } T \subset S, T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

Second, we define the P-reduced game $(S, v^{P,x})$ introduced by Funaki (1998).

$$v^{P,x}(T) = \begin{cases} v(N) - \sum_{i \in N \backslash S} x_i & \text{if } T = S, \\ v(T) & \text{if } T \subset S, T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

Third, we define the C-reduced game $(S, v^{C,x})$ introduced by Moulin (1985).

$$v^{C,x}(T) = \begin{cases} v\big(T \cup (N \backslash S)\big) - \sum_{i \in N \backslash S} x_i & \text{if } T \subseteq S, T \neq \emptyset, \\ 0 & \text{if } T = \emptyset. \end{cases}$$

We define the M-reduced game property satisfied by a solution ψ as follows:

M-Reduced Game Property (M-RGP) For any $v \in \Gamma^N$ and $S \subset N$, $S \neq \emptyset$, we have $\psi_i(N, v) = \psi_i(S, v^{M,\psi(N,v)})$ for all $i \in S$.

C-RGP and P-RGP are defined in a parallel manner.

Given a game $v \in \Gamma^N$, player $i \in N$ is called a null player if $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$, $i \notin S$. We list up axioms for a solution ψ.

[1]The notations of the three types include the capital letters M, P, C. The letter M refers to *Max* , P refers to *Projection* and C refers to *Complement* . Each term captures the characteristic of each type of reduced game.

Weak Strategic Invariance (WSI) For any $v \in \Gamma^N$ and $\beta \in \mathbb{R}^n$, we have $\psi(v+\beta) = \psi(v) + \beta$.

Efficiency (EFF) For any $v \in \Gamma^N$, $\sum_{i \in N} \psi_i(v) = v(N)$.

Null Player Property (NP) Take any $v \in \Gamma^N$. If $i \in N$ is a null player, then $\psi_i(v) = 0$.

Null Player Out (NPO) (Derks and Haller (1999)) Take any $v \in \Gamma^N$. If $i \in N$ is a null player, we have $\psi_j(N, v) = \psi_j(N \backslash \{i\}, v)$ for all $j \in N \backslash \{i\}$.

Equal Treatment Property (ETP) Take any $v \in \Gamma^N$ and $i,\ j \in N$. If $v(S \cup \{i\}) - v(S) = v(S \cup \{j\}) - v(S)$ for all $S \subseteq N \backslash \{i, j\}$, then $\psi_i(v) = \psi_j(v)$.

Linearity (LIN) For any $v,\ w \in \Gamma^N$, $\alpha,\ \beta \in \mathbb{R}$, we have $\psi(\alpha v + \beta w) = \alpha\psi(v) + \beta\psi(w)$.

Self Duality (SD) For any $v \in \Gamma^N$, $\psi(v^d) = \psi(v)$.

Self Anti-Duality (SAD) (Oishi and Nakayama (2009)) For any $v \in \Gamma^N$, $-\psi((-v)^d) = \psi(v)$.

Remark 1 (Theorem 2 of Oishi and Nakayama (2009)) *ϕ and PN satisfy SAD.* ∎

Note that the Shapley value satisfies WSI, EFF, NP, NPO, ETP, LIN , SD and SAD. The prenucleolus satisfies M-RGP, WSI, EFF, NP, NPO, ETP and SAD.[2] The CIS value satisfies P-RGP, WSI, EFF, ETP and LIN. The ENSC value satisfies C-RGP, WSI, EFF, ETP and LIN.

Before we move on to the next section, we give two lemmas.

Lemma 1 *For any $v \in \Gamma^N$,*

$$\phi(v + e_N) = \phi\Big(v + \sum_{S \subseteq N : S \neq \emptyset} e_S\Big).$$

[2] Kamijo and Kongo (2010) showed that ϕ satisfies NPO. We can show that PN satisfies NPO from NP and M-RGP.

Proof.

$$\begin{aligned}\phi(v+e_N) &= \phi(v)+\phi(e_N)\\ &= \phi(v)+\phi\Big(\sum_{S\subseteq N:S\neq\emptyset} e_S\Big)\\ &= \phi\Big(v+\sum_{S\subseteq N:S\neq\emptyset} e_S\Big),\end{aligned}$$

where the first and third equalities hold from LIN, and the second equality holds from ETP. □

Lemma 2 *For any $v \in \Gamma^N$,*

$$CIS(v) = ENSC(v^d).$$

We skip the proof since it is obvious from the definition of the two solutions and the dual game.

3 Coincidence condition for 3-person games

Let us remark that we fix the set of players N in this section. We first discuss the basis of the set of all games. The most famous one is unanimity games $(u_S)_{\emptyset\neq S\subseteq N}$ which was first introduced by Shapley (1953).

$$u_S(T) = \begin{cases} 1 \text{ if } S \subseteq T, \\ 0 \text{ otherwise.} \end{cases}$$

We now define one-leader games $(\bar{u}_S)_{\emptyset\neq S\subseteq N}$ as follows:

$$\bar{u}_S(T) = \begin{cases} 1 \text{ if } |T \cap S| = 1, \\ 0 \text{ otherwise.} \end{cases}$$

This game was introduced by Yokote et al. (2013).[3] In order to compare the two games, take a coalition $S \subseteq N$, $S \neq \emptyset$, and fix. Then, in unanimity game u_S, coalition $T \subseteq N$ obtains 1 if and only if T includes all players in S. On the other hand, in one-leader game $\bar{u}_T$, T obtains 1 if and only if T includes only one player in S.

Theorem 1 (Theorem 1 of Yokote et al. (2013)) *The set of games $(\bar{u}_S)_{\emptyset\neq S\subseteq N}$ is a basis of Γ^N.*

[3]From the definition, one-leader game is a simple game.

From Theorem 1, any game $v \in \Gamma^N$ can be expressed by a linear combination of $(\bar{u}_S)_{\emptyset \neq S \subseteq N}$. Let $d(S, v)$ be the coefficient of $\bar{u}_S$, $S \subseteq N$, $S \neq \emptyset$, in the linear combination. Namely,

$$v = \sum_{S \subseteq N : S \neq \emptyset} d(S, v)\bar{u}_S. \tag{1}$$

Remark 2 With respect to the basis, the following two equations hold:

$$\phi(\bar{u}_S) = \mathbf{0} \text{ for all } S \subseteq N, |S| \geq 2, \tag{2}$$

$$d(\{i\}, v) = \phi_i(v) \text{ for all } i \in N \text{ and } v \in \Gamma^N. \tag{3}$$

For the proof, see Theorem 2 of Yokote et al. (2013) and Theorem 4 of Yokote (2013). ∎

Let v^{Sh} denote the game $v - \phi(v)$. Namely, v^{Sh} is the game which satisfies

$$\begin{aligned} v^{Sh}(S) &= v(S) - \sum_{i \in S} \phi_i(v) \\ &= e(S, \phi(v), v), \end{aligned} \tag{4}$$

for all $S \subseteq N$, $S \neq \emptyset$. The game v^{Sh} is called the excess game of v with respect to $\phi(v)$. Note that $v^{Sh}(N) = 0$ and $\phi(v^{Sh}) = \mathbf{0}$. We can show that the game v^{Sh} satisfies $v^{Sh} = v - \sum_{i \in N} \phi_i(v) u_{\{i\}}$ from the following transformation: for any $S \subseteq N$, $S \neq \emptyset$,

$$\begin{aligned} v^{Sh}(S) &= v(S) - \sum_{i \in S} \phi_i(v) \\ &= v(S) - \sum_{i \in N} \phi_i(v) u_{\{i\}}(S) \\ &= \Big(v - \sum_{i \in N} \phi_i(v) u_{\{i\}}\Big)(S). \end{aligned}$$

Together with the fact that $u_{\{i\}} = \bar{u}_{\{i\}}$ for all $i \in N$, we have

$$v^{Sh} = v - \sum_{i \in N} \phi_i(v) \bar{u}_{\{i\}}. \tag{5}$$

As a result, v^{Sh} can be rewritten as follows:

$$\begin{aligned} v^{Sh} &= v - \phi(v) \\ &= v - \sum_{i \in N} \phi_i(v)\bar{u}_{\{i\}} \\ &= \sum_{S \subseteq N: S \neq \emptyset} d(S,v)\bar{u}_S - \sum_{i \in N} d(\{i\},v)\bar{u}_{\{i\}} \\ &= \sum_{S \subseteq N: |S| \geq 2} d(S,v)\bar{u}_S, \end{aligned} \tag{6}$$

where the second equality holds from equation (5) and the third equality holds from equation (3). The important implication of equation (6) is that we can express the game v^{Sh} by using only simple games. This simplicity is useful when we consider the coincidence condition.

Consider an arbitrary solution ψ which satisfies WSI. Then, we have

$$\psi(v) = \psi(v^{Sh} + \phi(v)) = \psi(v^{Sh}) + \phi(v).$$

It follows that

$$\phi(v) = \psi(v) \text{ if and only if } \psi(v^{Sh}) = \mathbf{0}. \tag{7}$$

That is, finding the coincidence condition is equivalent to investigating the payoff vector prescribed by the solution ψ to game v^{Sh}. We first give a necessary and sufficient condition under which the Shapley value coincides with the prenucleolus. Let us introduce an additional notation. We say that the family of coalitions $\mathscr{S} \subseteq 2^N$, $\emptyset \notin \mathscr{S}$, is balanced if the following condition is satisfied: there exists a collection of positive numbers $(\delta_S)_{\emptyset \neq S \subseteq N}$ such that for any $i \in N$,

$$\sum_{S: i \in S, S \in \mathscr{S}} \delta_S = 1.$$

From equation (7) and Kohlberg's (1971) theorem, we have the following proposition:

Proposition 1 *Let $v \in \Gamma^N$. Then, $\phi(v) = PN(v)$ if and only if for any $\alpha \in \mathbb{R}$, $\{S \subseteq N, S \neq \emptyset : v^{Sh}(S) \geq \alpha\} \neq \emptyset$ implies that the family of coalitions is balanced.*

By using this proposition, we precisely characterize the coincidence of the two solutions in the case of 3-person game.

Corollary 1 *Let $v \in \Gamma^N$, $N = \{1,2,3\}$, be a game which satisfies $v(\{i\}) = 0$ for all $i \in N$. Then, $\phi(v) = PN(v)$ if and only if v satisfies at least one of the following two conditions:*

Condition 1: $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$.

Condition 2: $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) = v(N)$.

For the proof, see Appendix. Since any 0-normalized game satisfies the condition $v(\{i\}) = 0$ and the two solutions satisfy WSI, Corollary 2 can be applied to any 3-person game. As we show in the proof, Condition 2 is equivalent to PS property introduced by Kar et al. (2009). So, Corollary 2 states that the coincidence region for 3-person 0-normalized game is exactly equal to the union of the set of all symmetric games, and the set of all games satisfying PS property.

We now give a remark on the coincidence between the Shapley value and the prenucleolus. We can show that the coincidence always holds in each game $\bar{u}_S$, $S \subseteq N$, $S \neq \emptyset$. To see this, we need the following result proved by Chang and Tseng (2011).

Remark 3 (Corollary 4 (iii) of Chang and Tseng (2011)) *Take any simple game $v \in \Gamma^N$. Then, $\phi(v) = \mathbf{0}$ implies that the family of coalitions $\{S \subseteq N : v(S) = 1\}$ is balanced.* ∎

Together with equation (2), for any $\alpha \in \mathbb{R}$ and $S \subseteq N$, $|S| \geq 2$, $\{T \subseteq N, T \neq \emptyset : \bar{u}_S(T) \geq \alpha\} \neq \emptyset$ implies that the family of coalitions is balanced. Then, Proposition 1 implies that $PN(\bar{u}_S) = \mathbf{0}$ for all $S \subseteq N$, $|S| \geq 2$. It follows that the coincidence is obtained in the game $\bar{u}_S$, $S \subseteq N$, $|S| \geq 2$. In the game $\bar{u}_S$, $S \subseteq N$, $|S| = 1$, we can show the coincidence from EFF and NP.

Next, we discuss the coincidence between the Shapley value and the CIS value, the ENSC value. We first give a note on the worth of each coalition in the game v^{Sh}. For any $v \in \Gamma^N$ and $S \subseteq N$, $S \neq \emptyset$, we have

$$\begin{aligned} v^{Sh}(S) &= \sum_{T \subseteq N : |T \cap S| = 1, |T| \geq 2} d(T, v) \\ &= \sum_{i \in S} \sum_{T \subseteq N \setminus S : T \neq \emptyset} d(\{i\} \cup T, v). \end{aligned}$$

We use this equation in the proof below.

Proposition 2 *For any $v \in \Gamma^N$, $\phi(v) = CIS(v)$ if and only if*

$$\sum_{T \subseteq N : i \in T, |T| \geq 2} (n - |T|) d(T) = \sum_{T \subseteq N \setminus \{i\}, |T| \geq 2} |T| \cdot d(T, v),$$

for all $i \in N$.

Proof. From equation (7), the condition $\phi(v) = CIS(v)$ is equivalent to

$$\begin{aligned}
CIS_i(v^{Sh}) &= v^{Sh}(\{i\}) - \frac{1}{n}\sum_{j\in N} v^{Sh}(\{j\}) = 0,\\
v^{Sh}(\{i\}) &= \frac{1}{n}\sum_{j\in N} v^{Sh}(\{j\}),\\
\sum_{T\subseteq N\setminus\{i\}:T\neq\emptyset} n\cdot d(\{i\}\cup T, v) &= \sum_{j\in N}\sum_{T\subseteq N\setminus\{j\}:T\neq\emptyset} d(\{j\}\cup T, v)\\
\sum_{T\subseteq N\setminus\{i\}:T\neq\emptyset} n\cdot d(\{i\}\cup T, v) &= \sum_{T\subseteq N:|T|\geq 2} |T|\cdot d(T,v),\\
\sum_{T\subseteq N:i\in T,|T|\geq 2} n\cdot d(T,v) &= \sum_{T\subseteq N:|T|\geq 2} |T|\cdot d(T,v),\\
\sum_{T\subseteq N:i\in T,|T|\geq 2} (n-|T|)d(T,v) &= \sum_{T\subseteq N\setminus\{i\},|T|\geq 2} |T|\cdot d(T,v),
\end{aligned}$$

for all $i \in N$. □

Proposition 3 *For any $v \in \Gamma^N$, $\phi(v) = ENSC(v)$ if and only if*

$$\sum_{T\subseteq N:i\in T,|T|=2} (n-2)d(T) = \sum_{T\subseteq N\setminus\{i\},|T|=2} 2\cdot d(T,v),$$

for all $i \in N$.

Proof. From equation (7), the condition $\phi(v) = ENSC(v)$ is equivalent to

$$\begin{aligned}
ENSC_i(v^{Sh}) &= -v^{Sh}(N\setminus\{i\}) + \frac{1}{n}\sum_{j\in N} v^{Sh}(N\setminus\{j\}) = 0,\\
n\cdot v^{Sh}(N\setminus\{i\}) &= \sum_{j\in N} v^{Sh}(N\setminus\{j\}),\\
\sum_{j\in N\setminus\{i\}} n\cdot d(\{i,j\},v) &= \sum_{j\in N}\sum_{k\in N\setminus\{j\}} d(\{j,k\},v),\\
\sum_{j\in N\setminus\{i\}} n\cdot d(\{i,j\},v) &= \sum_{T\subseteq N:|T|=2} 2\cdot d(T,v),\\
\sum_{T\subseteq N:i\in T,|T|=2} (n-2)d(T,v) &= \sum_{T\subseteq N\setminus\{i\}:|T|=2} 2\cdot d(T,v),
\end{aligned}$$

for all $i \in N$. □

We apply Propositions 2 and 3 to 3-person game.

Corollary 2 *Let $v \in \Gamma^N$ be a solution which satisfies $v(\{i\}) = 0$ for all $i \in N$. Then, the following three statements are equivalent:*

1: $\phi(v) = CIS(v)$.

2: $\phi(v) = ENSC(v)$.

3: $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$.

For the proof, see Appendix. Corollary 2 says that the set of games where the solutions coincide is equal to the set of all symmetric games.

Remark 4 Statements 1 and 2 of Corollary 2 are not equivalent in general. To see this, we give a game in which the Shapley value and the CIS value coincide but the Shapley value and the ENSC value do not.

Let $N = \{1,2,3,4,5\}$. Consider the game $\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}}$. From equation 2 and LIN, $\phi(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}}) = \mathbf{0}$. In order to calculate the CIS value and the ENSC value, the following worths are related:

$$\begin{array}{ll}
(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{1\}) = 1, & (\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{2,3,4,5\}) = 0, \\
(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{2\}) = 1, & (\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{1,3,4,5\}) = 0, \\
(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{3\}) = 1, & (\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{1,2,4,5\}) = 0, \\
(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{4\}) = 1, & (\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{1,2,3,5\}) = 1, \\
(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{5\}) = 1, & (\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}})(\{1,2,3,4\}) = 1.
\end{array}$$

It follows that $CIS(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}}) = \mathbf{0}$, while $ENSC(\bar{u}_{\{1,2,3\}} + \bar{u}_{\{4,5\}}) = (2/5, 2/5, 2/5, -3/5, -3/5)$.

∎

In the next section, we discuss the coincidence condition in some specific classes of games. We extend the result for 3-person game to n-person game by using reduced game property.

4 Coincidence condition in three classes of games

We divide this section into four parts. In subsections 4.1, 4.2 and 4.3, we define the class of airport games, bidder-collusion games and polluted river games, respectively. We give the coincidence condition in each class. In subsection 4.4, we discuss the result of this section.

4.1 Airport game

In this subsection, we investigate the coincidence condition in the class of airport games introduced by Littlechild and Owen (1973). A game $v \in \Gamma^N$ is called an airport game if there exists $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$ such that

$$v(S) = -\max_{i \in S} c_i \text{ for all } S \subseteq N, S \neq \emptyset.$$

Let $v_{A,c}$, $c = (c_i)_{i=1}^n$, $c_1 \geq c_2 \geq \cdots \geq c_n \geq 0$, denote this game, where A refers to *Airport*. The game $v_{A,c}$ is convex, and the prenucleolus and the nucleolus coincide.

The game $v_{A,c}$ captures the situation of constructing a runway in an airport. The value c_i represents the cost of accommodating i-th aircraft. As the size of an aircraft becomes larger, the cost becomes higher. Given a coalition, the cost of accommodating all aircrafts in the coalition depends on the largest size, so we take the maximum in the definition. In an airport, it is often the case that there are some aircrafts with the same type, which means the same size and the same cost. We capture the situation by allowing the possibility of equal costs.

In the class of airport games, the Shapley value and the nucleolus have been mainly investigated. The simple calculation methods of the two solutions were found by Littlechild and Owen (1973), and Littlechild (1974). Littlechild and Thompson (1977) applied the two solutions to the airport at Barmingham in England, and compared the results of game theoretic solutions with real data.

The main theorem of this section gives a necessary and sufficient condition under which the two solutions coincide. In the proof, we use the following proposition.

Proposition 4 (Littlechild and Owen (1973)) *Let $v_{A,c} \in \Gamma^N$ be an airport game. Then,*

$$\phi_i(v_{A,c}) = -\sum_{j=i}^{n-1} \frac{c_j - c_{j+1}}{j} - \frac{c_n}{n} \text{ for } j = 1, \cdots, n-1,$$
$$\phi_n(N, v_{A,c}) = -\frac{c_n}{n}.$$

Theorem 2 *Let $v_{A,c} \in \Gamma^N$, $n \geq 3$, be an airport game. Then, $\phi(v_{A,c}) = PN(v_{A,c})$ if and only if for any c_k, $3 \leq k \leq n$, $c_k = c_{k-1}$ or $c_k = 0$ holds.*

Proof. **If part:** From the assumption, one of the following two conditions holds:

1 $c_2 = \cdots = c_k$ and $c_{k+1} = \cdots = c_n = 0$ for some k, $2 \leq k \leq n-1$.

2 $c_2 = \cdots = c_n$.

If Condition 1 holds, then from the definition of the airport game, the players $k+1, \cdots, n$ are null players. From NPO, we only need to consider the subgame $(\{1, \cdots, k\}, v_{A,c})$, $c_2 = \cdots = c_k$, where $2 \leq k \leq n$. If $k = 2$, the coincidence trivially holds. Suppose that $3 \leq k \leq n$. We define $a := c_2 = \cdots = c_n$. Then, the game $v_{A,c}$ is given by

$$v_{A,c}(S) = \begin{cases} -c_1 & \text{if } 1 \in S, \\ -a & \text{if } 1 \notin S. \end{cases}$$

The 0-normalized game is given by

$$v^0_{A,c}(S) = \begin{cases} (|S|-1)a & \text{if } |S| \geq 2, \\ 0 & \text{if } |S| = 1. \end{cases} \tag{8}$$

From ETP, we have $\phi(v^0_{A,c}) = PN(v^0_{A,c})$. From WSI, we obtain the coincidence.

Only If part: We first consider the game $(N, v_{A,c})$, $N = \{1,2,3\}$. The 0-normalized game $v^0_{A,c}$ is given by

$$v^0_{A,c}(N) = c_2 + c_3, v^0_{A,c}(\{1,2\}) = c_2, v^0_{A,c}(\{1,3\}) = c_3, v^0_{A,c}(\{2,3\}) = c_3. \tag{9}$$

Since both solutions satisfy WSI, $\phi(v^0_{A,c}) = PN(v^0_{A,c})$. From Corollary 1, the vector c satisfies at least one of the following two conditions:

Condition 1: $c_2 = c_3$.

Condition 2: $c_3 = 0$.

It follows that the statement holds. We proceed by induction. Assume that the statement holds for $|N'| = n-1$, and we prove the case of $|N| = n$, $n \geq 4$. Let $x := \phi(v_{A,c}) = PN(v_{A,c})$. We consider the reduced game $(N \backslash \{n\}, v^{M,x}_{A,c})$. Note that from Proposition 4, $x_n = \phi_n(v_{A,c}) = -\frac{c_n}{n}$.

$$\begin{aligned} v^{M,x}_{A,c}(N \backslash \{n\}) &= v_{A,c}(N) + \frac{c_n}{n} = v_{A,c}(N \backslash \{n\}) + \frac{c_n}{n}, \\ v^{M,x}_{A,c}(S) &= \max\{v_{A,c}(S), v_{A,c}\big(S \cup \{n\}\big) + \frac{c_n}{n}\} \\ &= v_{A,c}(S) + \frac{c_n}{n} \text{ for all } S \subset N \backslash \{n\}, S \neq \emptyset. \end{aligned} \tag{10}$$

As a result,

$$(N \backslash \{n\}, v^{M,x}_{A,c}) = (N \backslash \{n\}, v_{A,c'}), \tag{11}$$

where $c'_i = c_i - \frac{c_n}{n}$ for $i = 1, \cdots, n-1$.

Claim 1

$$\phi_i(N\backslash\{n\}, v_{A,c'}) = \phi_i(N, v_{A,c}) \textit{ for all } i \in N\backslash\{n\}.$$

Proof. We first consider player i, $1 \leq i \leq n-2$. From Proposition 4,

$$\begin{aligned}
\phi_i(N, v_{A,c}) &= -\sum_{j=i}^{n-1} \frac{c_j - c_{j+1}}{j} - \frac{c_n}{n} \\
&= -\sum_{j=i}^{n-2} \frac{c_j - c_{j+1}}{j} - \frac{c_{n-1}}{n-1} + \frac{c_n}{n-1} - \frac{c_n}{n} \\
&= -\sum_{j=i}^{n-2} \frac{c_j - c_{j+1}}{j} - \frac{c_{n-1}}{n-1} + \frac{c_n}{n(n-1)}, \\
\phi_i(N\backslash\{n\}, v_{A,c'}) &= -\sum_{j=i}^{n-2} \frac{(c_j - \frac{c_n}{n}) - (c_{j+1} - \frac{c_n}{n})}{j} - \frac{c_{n-1} - \frac{c_n}{n}}{n-1} \\
&= -\sum_{j=i}^{n-2} \frac{c_j - c_{j+1}}{j} - \frac{c_{n-1}}{n-1} + \frac{c_n}{n(n-1)},
\end{aligned}$$

and the statement holds. Similarly,

$$\begin{aligned}
\phi_{n-1}(N, v_{A,c}) &= -\frac{c_{n-1} - c_n}{n-1} - \frac{c_n}{n} = -\frac{c_{n-1}}{n-1} + \frac{c_n}{n(n-1)}, \\
\phi_{n-1}(N\backslash\{n\}, v_{A,c'}) &= -\frac{c_{n-1} - \frac{c_n}{n}}{n-1} = -\frac{c_{n-1}}{n-1} + \frac{c_n}{n(n-1)},
\end{aligned}$$

which completes the proof. □

From equation (11), Claim 1 and M-RGP of the prenucleolus,

$$\phi_i(N\backslash\{n\}, v_{A,c'}) = x_i = PN_i(N\backslash\{n\}, v_{A,c'}) \text{ for all } i \in N\backslash\{n\}.$$

So, we can apply the induction hypothesis to game $(N\backslash\{n\}, v_{A,c'})$.

Case 1: Suppose that $c_2 - \frac{c_n}{n} = \cdots = c_k - \frac{c_n}{n}$ and $c_{k+1} - \frac{c_n}{n} = \cdots = c_{n-1} - \frac{c_n}{n} = 0$ for some k, $2 \leq k \leq n-2$. Then, from $c_{n-1} \geq c_n \geq 0$, we must have $c_n = 0$, which results in

$$c_2 = \cdots = c_k \text{ and } c_{k+1} = \cdots = c_{n-1} = c_n = 0,$$

and the statement holds.

Case 2: The remaining possibility is that $c_2 - \frac{c_n}{n} = \cdots = c_{n-1} - \frac{c_n}{n} > 0$, which implies $c_2 = \cdots = c_{n-1}$. Our goal is to prove that $c_{n-1} = c_n$ or $c_n = 0$ holds. Assume not, that is, $c_{n-1} > c_n > 0$. We derive a contradiction. By letting $a := c_2 = \cdots = c_{n-1}$, the worths of coalitions are expressed as follows:

$$v_{A,c}(S) = \begin{cases} -c_1 & \text{if } 1 \in S, \\ -a & \text{if } 1 \notin S, S \neq \{n\}, \\ -c_n & \text{if } S = \{n\}. \end{cases} \tag{12}$$

We calculate the Shapley value. From Proposition 4,

$$\begin{aligned} \phi_n(N, v_{A,c}) &= -\frac{c_n}{n}, \\ \phi_j(N, v_{A,c}) &= -\Big\{\frac{a - c_n}{n-1} + \frac{c_n}{n}\Big\} \text{ for } j = 2, \cdots, n-1. \end{aligned}$$

From EFF,

$$\phi_1(N, v_{A,c}) = -c_1 + \frac{c_n}{n} + (n-2)\Big\{\frac{a - c_n}{n-1} + \frac{c_n}{n}\Big\}.$$

Since we assume that $a > c_n > 0$, $\phi_i(N, v) < 0$ for all $i \in N$. As a result, the excess of each coalition increases as the number of players in the coalition increases. Together with the fact that $0 > \phi_n(N, v_{A,c}) > \phi_{n-1}(N, v_{A,c})$, we have

$$\{S \subset N : 1 \in S, e(S, x, v_{A,c}) \geq e(T, x, v_{A,c}) \text{ for all } T \subset N, 1 \in T\} = \big\{N \backslash \{n\}\big\}.$$

Similarly,

$$\begin{aligned} &\big\{S \subset N : 1 \notin S, S \neq \{n\}, e(S, x, v_{A,c}) \geq e(T, x, v_{A,c}) \text{ for all } T \subset N, 1 \notin T, T \neq \{n\}\big\} \\ &= \big\{N \backslash \{1\}\big\}. \end{aligned}$$

As a result, the set of coalitions with maximum excess consists of some of the following four coalitions: $N \backslash \{n\}$, $N \backslash \{1\}$, $\{n\}$ and $\{N\}$.

We calculate the excess of each coalition.

$$
\begin{aligned}
e(N\backslash\{n\}, x, v_{A,c}) =& - c_1 - \Big[-c_1 + \frac{c_n}{n} + (n-2)\Big\{\frac{a-c_n}{n-1} + \frac{c_n}{n}\Big\}\Big] \\
& - (n-2)\Big[-\Big\{\frac{a-c_n}{n-1} + \frac{c_n}{n}\Big\}\Big] \\
=& - \frac{c_n}{n}, \\
e(\{n\}, x, v_{A,c}) =& - c_n + \frac{c_n}{n} \\
=& - \frac{(n-1)c_n}{n}, \\
e(N\backslash\{1\}, x, v_{A,c}) =& - a + \frac{c_n}{n} + (n-2)\Big\{\frac{a-c_n}{n-1} + \frac{c_n}{n}\Big\} \\
=& - \frac{a}{n-1} + \frac{c_n}{n(n-1)}.
\end{aligned}
$$

Since we assume that $c_n > 0$, we have $e(N\backslash\{n\}, x, v_{A,c}) > e(\{n\}, x, v_{A,c})$. In addition,

$$
\begin{aligned}
e(N\backslash\{n\}, x, v_{A,c}) - e(N\backslash\{1\}, x, v_{A,c}) &= -\frac{c_n}{n} + \frac{a}{n-1} - \frac{c_n}{n(n-1)} \\
&= \frac{n(a-c_n)}{n(n-1)} > 0,
\end{aligned}
$$

where the last inequality holds from $a > c_n$ and $n \geq 4$. We also have

$$
e(N, x, v_{A,c}) = 0 > e(N\backslash\{n\}, x, v_{A,c}),
$$

where the last equality holds from $c_n > 0$. Together with equation (4), the above inequalities imply

$$
\big\{S \subseteq N, S \neq \emptyset : v^{Sh}_{A,c}(S) \geq v^{Sh}_{A,c}(N\backslash\{n\})\big\} = \big\{N\backslash\{n\}, N\big\},
$$

which violates Proposition 1, and we obtain the desired contradiction. □

Next, we give the coincidence condition between the Shapley value and the CIS value, the ENSC value.

Theorem 3 *Let $v_{A,c} \in \Gamma^N$ be an airport game. Then, the following statements are equivalent:*

1: $\phi(v_{A,c}) = CIS(v_{A,c})$.

2: $\phi(v_{A,c}) = ENSC(v_{A,c})$.

3: $c_2 = \cdots = c_n$.

Proof. If 3 holds, the 0-normalized game is expressed by equation (8). From ETP of the CIS value and the ENSC value, $3 \Rightarrow 1$ and $3 \Rightarrow 2$ can be obtained. We prove $1 \Rightarrow 3$ and $2 \Rightarrow 3$.

Proof of $1 \Rightarrow 3$**:** Let $x := \phi(v_{A,c}) = CIS(v_{A,c})$. Consider the game $(N, v_{A,c})$, $N = \{1, 2, 3\}$. The 0-normalized game is given by equation (9). From Corollary 2, we have

$$c_2 = c_3,$$

and the statement holds. Assume that the statement holds for player set $|N'| = n - 1$, and we prove the case of $|N| = n$, $n \geq 4$. We consider the P-reduced game $(N\backslash\{n\}, v_{A,c}^{P,x})$.

$$\begin{aligned} v_{A,c}^{P,x}(N\backslash\{n\}) = v_{A,c}(N) + \frac{c_n}{n} = v_{A,c}(N\backslash\{n\}) + \frac{c_n}{n}, \\ v_{A,c}^{P,x}(S) = v_{A,c}(S) \text{ for all } S \subset N\backslash\{n\}, S \neq \emptyset. \end{aligned} \tag{13}$$

By letting $c' := (c_1 - \frac{c_n}{n}, \cdots, c_n - \frac{c_n}{n})$, we have

$$\begin{aligned} \phi_i(N\backslash\{n\}, v_{A,c}^{P,x}) &= \phi_i\big(N\backslash\{n\}, v_{A,c} + \frac{c_n}{n} e_{N\backslash\{n\}}\big) \\ &= \phi_i\Big(N\backslash\{n\}, v_{A,c} + \frac{c_n}{n} \sum_{S \subseteq N\backslash\{n\}: S \neq \emptyset} e_S\Big) \\ &= \phi(N\backslash\{n\}, v_{A,c}^{M,x}) \\ &= \phi_i(N\backslash\{n\}, v_{A,c'}) \\ &= \phi_i(N, v_{A,c}) \text{ for all } i \in N\backslash\{n\}, \end{aligned}$$

where the first equality holds from equation (13), the second equality holds from Lemma 1, the third equality holds from equation (10), the fourth equality holds from equation (11) and the last equality holds from Claim 1. Together with P-RGP of the CIS value, we have

$$\phi_i(N\backslash\{n\}, v_{A,c}^{P,x}) = x_i = CIS_i(N\backslash\{n\}, v_{A,c}^{P,x}) \text{ for all } i \in N\backslash\{n\}.$$

Since $v_{A,c}^{P,x} = v_{A,c} + \frac{c_n}{n} e_{N\backslash\{n\}}$, together with the above equation and LIN of both solutions, we have

$$\begin{aligned} &\phi(N\backslash\{n\}, v_{A,c}) + \phi\big(N\backslash\{n\}, \frac{c_n}{n} e_{N\backslash\{n\}}\big) \\ =&CIS(N\backslash\{n\}, v_{A,c}) + CIS\big(N\backslash\{n\}, \frac{c_n}{n} e_{N\backslash\{n\}}\big). \end{aligned} \tag{14}$$

From ETP of both solutions,

$$\phi\big(N\backslash\{n\}, \frac{c_n}{n} e_{N\backslash\{n\}}\big) = CIS\big(N\backslash\{n\}, \frac{c_n}{n} e_{N\backslash\{n\}}\big).$$

As a result, equation (14) reduces to

$$\phi(N\backslash\{n\}, v_{A,c}) = CIS(N\backslash\{n\}, v_{A,c}).$$

From the induction hypothesis,

$$c_2 = \cdots = c_{n-1}.$$

By letting $a := c_2 = \cdots = c_{n-1}$, the game $v_{A,c}$ can be rewritten by equation (12). From Proposition 4, $\phi_n(N, v_{A,c}) = -\frac{c_n}{n}$. On the other hand,

$$\begin{aligned} CIS_n(N, v_{A,c}) &= -c_n + \frac{-c_1 + c_1 + c_n + (n-2)a}{n} \\ &= \frac{-(n-1)c_n + (n-2)a}{n}. \end{aligned}$$

It follows that $-(n-1)c_n + (n-2)a = -c_n$. From $n \geq 4$, we have $a = c_n$.

Proof of $2 \Rightarrow 3$**:** By following the same argument of the proof of $1 \Rightarrow 3$, we have the result for 3-person game. We proceed by induction. Consider the game $(N, v_{A,c})$, $n \geq 4$, and let $x := \phi(N, v_{A,c}) = ENSC(N, v_{A,c})$. From equation (10), we have $(N\backslash\{n\}, v_{A,c}^{M,x}) = (N\backslash\{n\}, v_{A,c}^{C,x})$. From Claim 1 and C-RGP of the ENSC value,

$$\phi_i(N\backslash\{n\}, v_{A,c'}) = ENSC_i(N\backslash\{n\}, v_{A,c'}) \text{ for all } i \in N\backslash\{n\},$$

where $c' = (c_1 - \frac{c_n}{n}, \cdots, c_n - \frac{c_n}{n})$. From the induction hypothesis, $c_2 - \frac{c_n}{n} = \cdots = c_{n-1} - \frac{c_n}{n}$, which implies $c_2 = \cdots = c_{n-1}$. Then, the game $v_{A,c}$ can be written by equation (12). Proposition 4 implies $\phi_n(N, v_{A,c}) = -\frac{c_n}{n}$, and $ENSC_n(N, v_{A,c}) = -\frac{a}{n}$. As a result, we obtain $a = c_n$. □

Remark 5 From Theorem 3, for any airport game $v_{A,c}$, $c_2 = \cdots = c_n$ implies that $CIS(v_{A,c}) = ENSC(v_{A,c})$. We can prove that the inverse logic also holds. To see this, assume that $CIS(v_{A,c}) = ENSC(v_{A,c})$. The payoff of player 1 is given as follows:

$$CIS_1(v_{A,c}) = -c_1 + \frac{1}{n} \sum_{j=2}^{n} c_j,$$

$$ENSC_1(v_{A,c}) = -c_1 + \frac{(n-1)c_2}{n}.$$

It follows that $\sum_{j=2}^{n} c_j = (n-1)c_2$. Since $c_2 \geq c_j$ for all $j = 2, \cdots, n$, we obtain $c_2 = \cdots = c_n$. ∎

4.2 Bidder-collusion game

The coincidence condition for airport game can also be applied to bidder collusion games introduced by Graham et al. (1990). A game $v \in \Gamma^N$ is called a bidder collusion game if there exists $v_1 \geq v_2 \geq \cdots \geq v_n \geq 0$ such that

$$v(S) = \begin{cases} v_1 & \text{if } S = N, \\ v_1 - \max_{j \notin S} v_j & \text{if } 1 \notin S, \\ 0 & \text{otherwise.} \end{cases}$$

Let $v_{B,v}$, $v = (v_i)_{i=1}^n$, $v_1 \geq v_2 \geq \cdots \geq v_n \geq 0$, denote this game, where B refers to *Bidder*. The game $v_{B,v}$ captures the English auction with complete information in which the following assumptions are satisfied:

1: The seller begins with zero price.

2: If some bidders drop out of the auction at the same time, the bidder with the lowest index obtains the object.

The word 'bidder collusion' refers to the coalitional behavior among bidders. If coalition S forms, the coalition behaves as one bidder, and the 'bidder' S drops out if the price is equal to the maximum valuation among all bidders in S. If bidders make a coalition, they might obtain the object with lower price and yield net surplus, since making coalition alleviates the price competition among bidders. The characteristic function captures the net surplus. The objective here is to investigate the way of distributing the surplus.

The coincidence condition in the class of bidder collusion games is similar to that of airport games. The reason is that the two games are related through the following relationship.[4,5]

Proposition 5 (Proposition 1 of Oishi and Nakayama) *Let $x \in \mathbb{R}^n$, $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$. Then,*

$$(-v_{A,x})^d = v_{B,x} \text{ and } (-v_{B,x})^d = v_{A,x}.$$

Recall that the Shapley value and the prenucleolus satisfy SAD. As a result, the following corollary of Theorem 2 holds.

[4]The relationship was called anti-dual by Oishi and Nakayama (2009). As they showed, the anti-dual preserves the convexity of a game. So, the bidder-collusion game is also a convex game.

[5]Although Oishi and Nakayama (2009) considered the case $x_1 > \cdots > x_n > 0$, the result here can be similarly obtained.

Corollary 3 *Let $v_{B,v} \in \Gamma^N$, $n \geq 3$, be a bidder collusion game. Then, $\phi(v_{B,v}) = PN(v_{B,v})$ if and only if for any v_k, $3 \leq k \leq n$, $v_k = v_{k-1}$ or $v_k = 0$ holds.*

Proof. From SAD, $\phi(v_{B,x}) = PN(v_{B,x})$ if and only if $-\phi((-v_{B,x})^d) = -PN((-v_{B,x})^d)$. From Proposition 5, this equation holds if and only if $-\phi(v_{A,x}) = -PN(v_{A,x})$. From Theorem 2, the proof is completed. □

The corollary of Theorem 3 is given as follows:

Corollary 4 *Let $v_{B,v} \in \Gamma^N$ be an airport game. Then, the following statements are equivalent:*

1: $\phi(v_{B,v}) = CIS(v_{B,v})$.

2: $\phi(v_{B,v}) = ENSC(v_{B,v})$.

3: $c_2 = \cdots = c_n$.

Proof. **Proof of** 1 ⇔ 3**:** From Proposition 5, $\phi(v_{B,x}) = CIS(v_{B,x})$ if and only if $\phi\big((-v_{A,x})^d\big) = CIS\big((-v_{A,x})^d\big)$. Since $(-v_{A,x})^d = -(v_{A,x})^d$, from LIN of both solutions, $-\phi\big((v_{A,x})^d\big) = -CIS\big((v_{A,x})^d\big)$. From SD of the Shapley value and Lemma 2, $\phi(v_{A,x}) = ENSC(v_{A,x})$. From Theorem 3, the condition is equivalent to statement 3, which completes the proof.

Proof of 2 ⇔ 3**:** We can follow the same argument of the proof of 1 ⇔ 3. □

4.3 Polluted river game

In this subsection, we consider the polluted river games introduced by Ni and Wang (2007). A game $v \in \Gamma^N$, $N = \{1, \cdots, n\}$, is called a polluted river game if there exists $(c_1, \cdots, c_n) \in \mathbb{R}^n_+$ such that for any $S \subseteq N$, $S \neq \emptyset$,

$$v(S) = -\sum_{i=\min S}^{n} c_i,$$

where $\min S = \{i \in S : i \leq j \text{ for all } j \in S\}$. Let $v_{P,c}$ denote this game, where P refers to *Polluted*. We explain the cost allocation problem behind the above formulation. Consider a river which is divided into n segments, and n firms which are located in each segment. We assume that the firm with lower index is located in the upstream of the river. The parameter c_i, $i \in N$, represents the cost to clean the pollution in each segment. In this setting, we consider the cost-sharing problem among n firms. The characteristic function

$v_{P,c}$ captures the fact that each coalition S is responsible for the pollution of all segments in its downstream.

We use the following lemma.

Lemma 3 *For any* $(c_1, \cdots, c_n) \in \mathbb{R}^n_+$, *let* $c' \in \mathbb{R}^n_+$ *denote the following vector:*

$$c'_k = \sum_{i=k}^{n} c_k \text{ for all } k = 1, \cdots, n. \tag{15}$$

Then, we have

$$v_{P,c}(S) = v_{A,c'}(S) \text{ for all } S \subseteq N, S \neq \emptyset.$$

Proof. For any $S \subseteq N$, $S \neq \emptyset$, we have

$$v_{P,c}(S) = -\sum_{i=\min S}^{n} c_i = -\max_{i \in S} c'_i = v_{A,c'}(S),$$

which completes the proof. □

The coincidence condition between the Shapley value and the prenucleolus is given in the next theorem.

Theorem 4 *Let* $v_{P,c} \in \Gamma^N$, $n \geq 3$, *be a polluted river game. Then,* $\phi(v_{P,c}) = PN(v_{P,c})$ *if and only if there exists at most one* c_k, $2 \leq k \leq n$ *such that* $c_k > 0$.

Proof. From Lemma 3, $\phi(v_{P,c}) = PN(v_{P,c})$ if and only if $\phi(v_{P,c'}) = PN(v_{P,c'})$, where c' is given by equation (15). From Theorem 2, the coincidence holds if and only if for any c'_k, $3 \leq k \leq n$, $c'_k = c'_{k-1}$ or $c'_k = 0$ holds. That is, for any c_k, $3 \leq k \leq n$, $c_{k-1} = 0$ or $c_k = \cdots = c_n = 0$ holds. The equivalent condition is given in the statement. □

Next, we give the coincidence condition between the Shapley value and the CIS value, the ENSC value.

Theorem 5 *Let* $v_{P,c} \in \Gamma^N$, $n \geq 3$, *be a polluted river game. Then, the following three statements are equivalent:*

1: $\phi(v_{P,c}) = CIS(v_{P,c})$.

2: $\phi(v_{P,c}) = ENSC(v_{P,c})$.

3: $c_2 = \cdots = c_{n-1} = 0$.

Proof. From Lemma 3, we only need to consider the coincidence in the game $v_{A,c'}$, where c' is given by equation (13). From Theorem 3, the following three conditions are equivalent:

1′: $\phi(v_{A,c'}) = CIS(v_{A,c'})$.

2′: $\phi(v_{A,c'}) = ENSC(v_{A,c'})$.

3′: $c'_2 = \cdots = c'_n$.

Condition 3′ is equivalent to Condition 3 in the statement, and the proof is completed. □

Remark 6 We can also consider the anti-dual game of polluted river game. Let $c = (c_1, \cdots, c_n) \in \mathbb{R}^n_+$ be given. Then,

$$\begin{aligned} -v^d_{P,c}(S) &= -v_{P,c}(N) + v_{P,c}(N \backslash S) \\ &= \sum_{i=1}^{n} c_i - \sum_{i=\min N\backslash S}^{n} c_i \\ &= \begin{cases} 0 & \text{if } 1 \notin S, \\ \sum_{k=1}^{t} c_k & \text{if there exists } t,\ 1 \le t \le n-1, \text{ such that} \\ & i \in S \text{ for all } i \le t \text{ and } t+1 \notin S. \end{cases} \end{aligned}$$

From SAD of the Shapley value and the prenucleolus, the coincidence condition with the prenucleolus in the above class of games is the same as the condition of Theorem 4. Similarly, from duality between the CIS value and the ENSC value, the coincidence condition with the CIS value, the ENSC value is also the same as the condition of Theorem 5. ∎

4.4 Discussion

The results of this section indicate that in each class of games, different solutions seldom coincide. All theorems state that under the coincidence condition, at most two different positive parameters appear. In other words, if there are three different positive parameters, then the Shapley value does not coincide with another solution. It follows that, in the usual economic situations, the Shapley value is not desirable from the viewpoint of other solutions. We can also regard the results as the part of impossibility theorem. By restricting possible parameters in each class, we can prove the impossibility of coincidence between different solutions. For example, if we formulate the class of airport games with parameters given by $c_1 > c_2 > \cdots > c_n > 0$,

$n \geq 3$, then there exists no game where different solutions analyzed here coincide.

In each class of games, the set of games where the Shapley value and the CIS value, the ENSC value coincide is a proper subset of the set of games where the Shapley value and the prenucleolus coincide. In other words, if the Shapley value coincides with the CIS value or the ENSC value, then the Shapley value coincides with the prenucleolus, but the inverse logic does not necessarily hold. This result seems counter-intuitive if we recall the definition of the solutions. The CIS value and the ENSC value can be calculated by addition and multiplication like the Shapley value, and the three solutions share the strong axiom, LIN. On the other hand, the prenucleolus, which is a lexicographic minimization of the excess, is derived from a totally different calculation.

In order to discover the reason, we focus on the game where the coincidence condition with the prenucleolus is satisfied but the condition with the CIS value, the ENSC value is not satisfied. From the theorems in the previous section, we can know that there is at least one 0 parameter in such a game. Equivalently, there is at least one null player. This observation enables us to explain the question about the coincidence region from null player property. Recall that the Shapley value and the prenucleolus satisfy the axiom, but the CIS value and the ENSC value do not. As a result, when there is a null player, it is more likely that the Shapley value coincides with the prenucleolus.

5 Concluding remarks

In this paper, we focused on the relationship between one-point solutions. One possible extension is to investigate the relationship between the Shapley value and set-valued solutions. As an example, consider the Core which is defined by

$$C(v) = \Big\{ x \in PI(v) : \sum_{i \in S} x_i \geq v(S) \text{ for all } S \subseteq N, S \neq \emptyset \Big\},$$

for all $v \in \Gamma^N$. Then, equation (7) is rearranged in the following way:

$$\phi(v) \in C(v) \text{ if and only if } \mathbf{0} \in C(v^{Sh}).$$

Based on the simple expression of v^{Sh} given in equation (6), there is a possibility that the new condition under which the Shapley value belongs to the Core is obtained. For example, the condition for 3-person game is given as follows:

Remark 7 *Let $v \in \Gamma^N$, $N = \{1, 2, 3\}$, be a game which satisfies $v(\{i\}) = 0$ for all $i \in N$ and $v(\{1, 2\}) \geq v(\{1, 3\}) \geq v(\{2, 3\})$. Then, $\phi(v) \in C(v)$ if and only if the following inequality holds:*

$$v(12) + v(13) + 4v(23) \leq \min\{4v(N), 2v(12) + 2v(13) + 2v(23) + 2v(N)\}.$$

∎

For the proof, see Appendix. The extension of the above condition to the general case is left as an open question.

We showed the coincidence in some classes of games. The key in the proof is the induction on the number of players. If we consider the reduced game which 'eliminates' the last player, then the Shapley value satisfies reduced game property discussed in this paper.[6] If we can find another class of games where the Shapley value satisfies reduced game property, then the proof of this paper can be applied to prove the coincidence condition in the class.

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Appendix

In this appendix, we prove Corollaries 1, 2 and Remark 7. In the proof, we use the following theorem given by Yokote et al. (2013).

Theorem 6 (Theorem 2 of Yokote et al. (2013)) *Let $v \in \Gamma^N$. Then, we have*

$$d(S,v) = (-1)^{|S|-1} \sum_{T:S \subseteq T} \frac{1}{|T|} D(T,v),$$

for all $S \subseteq N, S \neq \emptyset$. In particular,

$$d(\{i\},v) = \sum_{T \subseteq N: i \in T} \frac{1}{|T|} D(T,v) = \phi_i(N,v),$$

for all $i \in N$.

By applying the theorem to the game $v \in \Gamma^N$, $N = \{1,2,3\}$, we have

$$d(N,v) = -\frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{3} + \frac{v(N)}{3}. \tag{16}$$

Similarly, we have

$$\begin{aligned} d(\{1,2\},v) &= -\frac{v(\{1,2\})}{6} + \frac{v(\{1,3\})}{3} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3}, \\ d(\{1,3\},v) &= \frac{v(\{1,2\})}{3} - \frac{v(\{1,3\})}{6} + \frac{v(\{2,3\})}{3} - \frac{v(N)}{3}, \\ d(\{2,3\},v) &= \frac{v(\{1,2\})}{3} + \frac{v(\{1,3\})}{3} - \frac{v(\{2,3\})}{6} - \frac{v(N)}{3}. \end{aligned} \tag{17}$$

Proof of Corollary 1. **If part:** Under Condition 1, the game is symmetric, so the coincidence is obvious from ETP.

Suppose that Condition 2 holds. Then, we have

$$v(\{1\}) - v(\emptyset) + v(N) - v(\{2,3\}) = v(\{1,2\}) + v(\{1,3\}),$$
$$v(\{1,2\}) - v(\{2\}) + v(\{1,3\}) - v(\{3\}) = v(\{1,2\}) + v(\{1,3\}).$$

Namely, for any coalition S such that $1 \notin S$, the sum of marginal contributions of player 1 to coalition S and to coalition $N \backslash (S \cup \{1\})$ is the same. The same condition holds for players 2 and 3. This property is known as PS property introduced by Kar, Mitra and Mutuswami (2009). As they proved, under PS property, the Shapley value and the prenucleolus coincide.

Only If part: Suppose not. Then, both Conditions 1 and 2 do not hold. Assume, without loss of generality, that $v(\{1,2\}) \geq v(\{1,3\}) \geq v(\{2,3\})$. Since Condition 1 does not hold, we have either $v(\{1,2\}) > v(\{1,3\}) \geq v(\{2,3\})$ or $v(\{1,2\}) = v(\{1,3\}) > v(\{2,3\})$.

Case 1: $v(\{1,2\}) > v(\{1,3\}) \geq v(\{2,3\})$.

The following two inequalities hold:

$$-2v(\{2,3\}) \geq -2v(\{1,3\}) > -2v(\{1,2\}), \tag{18}$$
$$v(\{1,2\}) + v(\{1,3\}) \geq v(\{1,2\}) + v(\{2,3\}) > v(\{1,3\}) + v(\{2,3\}). \tag{19}$$

By adding $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})$ to each side of equation (19), we have

$$\begin{aligned} &2v(\{1,2\}) + 2v(\{1,3\}) + v(\{2,3\}) \\ \geq &2v(\{1,2\}) + v(\{1,3\}) + 2v(\{2,3\}) \\ > &v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}). \end{aligned} \tag{20}$$

The addition of equations (18) and (20) implies

$$\begin{aligned} &2v(\{1,2\}) + 2v(\{1,3\}) - v(\{2,3\}) \\ \geq &2v(\{1,2\}) - v(\{1,3\}) + 2v(\{2,3\}) \\ > &- v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}). \end{aligned}$$

By dividing by 6, together with equation (17), we have

$$d(\{2,3\}, v) \geq d(\{1,3\}, v) > d(\{1,2\}, v).$$

It follows that

$$d(\{1,3\},v)+d(\{2,3\},v) > d(\{1,2\},v)+d(\{2,3\},v) \geq d(\{1,2\},v)+d(\{1,3\},v). \tag{21}$$

Recall that we can calculate the worths of coalitions in v^{Sh} as follows:

$$\begin{aligned} v^{Sh}(\{1\}) &= d(\{1,2\},v) + d(\{1,3\},v) + d(N,v), \\ v^{Sh}(\{2\}) &= d(\{1,2\},v) + d(\{2,3\},v) + d(N,v), \\ v^{Sh}(\{3\}) &= d(\{1,3\},v) + d(\{2,3\},v) + d(N,v), \\ v^{Sh}(\{1,2\}) &= d(\{1,3\},v) + d(\{2,3\},v), \\ v^{Sh}(\{1,3\}) &= d(\{1,2\},v) + d(\{2,3\},v), \\ v^{Sh}(\{2,3\}) &= d(\{1,2\},v) + d(\{1,3\},v), \\ v^{Sh}(N) &= 0. \end{aligned} \tag{22}$$

From equation (21), we have

$$\begin{aligned} &v^{Sh}(\{1,2\}) > v^{Sh}(\{1,3\}) \geq v^{Sh}(\{2,3\}), \\ &v^{Sh}(\{3\}) > v^{Sh}(\{2\}) \geq v^{Sh}(\{1\}). \end{aligned}$$

Since Condition 2 does not hold, $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) \neq v(N)$. Assume that $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) > v(N)$. Then, we have

$$-v(\{1,2\}) - v(\{1,3\}) - v(\{2,3\}) + v(N) < 0,$$

which implies, together with equation (16), $d(N,v) < 0$. In this case,

$$\{S \subseteq N, S \neq \emptyset : v^{Sh}(S) \geq v^{Sh}(\{1,2\})\} = \{1,2\},$$

which violates Proposition 1.

If we assume that $v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\}) < v(N)$, we have $d(N,v) > 0$. In this case,

$$\{S \subseteq N, S \neq \emptyset : v^{Sh}(S) \geq v^{Sh}(\{3\})\} = \{3\},$$

which again violates Proposition 1.

Case 2: $v(\{1,2\}) = v(\{1,3\}) > v(\{2,3\})$

By following the same calculation of Case 1, we have

$$\begin{aligned} &v^{Sh}(\{1,2\}) = v^{Sh}(\{1,3\}) > v^{Sh}(\{2,3\}), \\ &v^{Sh}(\{3\}) = v^{Sh}(\{2\}) > v^{Sh}(\{1\}). \end{aligned}$$

If we assume that $v(\{1,2\})+v(\{1,3\})+v(\{2,3\}) > v(N)$, we have $d(N,v) < 0$, which results in

$$\{S \subseteq N, S \neq \emptyset : v^{Sh}(S) \geq v^{Sh}(\{1,2\})\} = \big\{\{1,2\},\{1,3\}\big\}.$$

If we assume that $v(\{1,2\})+v(\{1,3\})+v(\{2,3\}) < v(N)$, we have $d(N,v) > 0$, which results in

$$\{S \subseteq N, S \neq \emptyset : v^{Sh}(S) \geq v^{Sh}(\{2\})\} = \big\{\{2\},\{3\}\big\}.$$

Both results violate Proposition 1. □

Proof of Corollary 2. We prove $1 \Leftrightarrow 3$. From Proposition 2, we have

$$\begin{aligned}
d(\{1,2\},v) + d(\{1,3\},v) &= 2d(\{2,3\},v),\\
d(\{1,2\},v) + d(\{2,3\},v) &= 2d(\{1,3\},v),\\
d(\{1,3\},v) + d(\{2,3\},v) &= 2d(\{1,2\},v).
\end{aligned}$$

By substituting equations (17) to the above equations, we have

$$\begin{aligned}
v(\{1,2\}) + v(\{1,3\}) &= 2v(\{2,3\}),\\
v(\{1,2\}) + v(\{2,3\}) &= 2v(\{1,3\}),\\
v(\{1,3\}) + v(\{2,3\}) &= 2v(\{1,2\}).
\end{aligned}$$

These equations hold if and only if $v(\{1,2\}) = v(\{1,3\}) = v(\{2,3\})$. The proof of $2 \Leftrightarrow 3$ can be similarly obtained. □

Proof of Remark 7. The condition $\phi(v) \in C(v^{Sh})$ holds if and only if the following inequalities holds:

$$\begin{aligned}
v^{Sh}(\{1\}) &= d(\{1,2\},v) + d(\{1,3\},v) + d(N,v) \leq 0,\\
v^{Sh}(\{2\}) &= d(\{1,2\},v) + d(\{2,3\},v) + d(N,v) \leq 0,\\
v^{Sh}(\{3\}) &= d(\{1,3\},v) + d(\{2,3\},v) + d(N,v) \leq 0,\\
v^{Sh}(\{1,2\}) &= d(\{1,3\},v) + d(\{2,3\},v) \leq 0,\\
v^{Sh}(\{1,3\}) &= d(\{1,2\},v) + d(\{2,3\},v) \leq 0,\\
v^{Sh}(\{2,3\}) &= d(\{1,2\},v) + d(\{1,3\},v) \leq 0.
\end{aligned} \tag{23}$$

From the assumption, the game v satisfies $v(\{1,2\}) \geq v(\{1,3\}) \geq v(\{2,3\})$. By following the same argument which derived equation (21), we obtain

$$d(\{1,3\},v)+d(\{2,3\},v) \geq d(\{1,2\},v)+d(\{2,3\},v) \geq d(\{1,2\},v)+d(\{1,3\},v).$$

As a result, equation (23) is equivalent to

$$d(\{1,3\},v) + d(\{2,3\},v) \leq \min\{-d(N,v), 0\}$$

By substituting equations (16), (17) and rearranging, we have

$$\begin{aligned}&4v(\{1,2\}) + v(\{1,3\}) + v(\{2,3\})\\ &\leq \min\{4v(N), 2v(\{1,2\}) + 2v(\{1,3\}) + 2v(\{2,3\}) + 2v(N)\}.\end{aligned}$$